Theoretische Physik V
- Quantenmechanik II (WS 2018/2019) -

Übungsblatt 2

Ausgabe 23.10.18 – Abgabe 08.11.18 – Besprechung 09.11.18

▷ **Aufgabe 1 (Witch-path detection)** (20 scores)

We consider a Young double slit experiment for spin-$\frac{1}{2}$ particle. We assume some magnetic field behind one of the slits – the slit on the right side, say. As we shall see the magnetic field – assumed to have no impact on the motional state of the atoms – may have a severe impact on the interference pattern of the double slit.

The particles impinge in some definite spin state $|\uparrow_z\rangle$. Behind the double slit the state vector reads

$$|\Psi\rangle = \frac{|l\rangle \otimes |\uparrow_z\rangle + |r\rangle \otimes |\uparrow_a\rangle}{\sqrt{2}}, \tag{1}$$

where $|l\rangle$ ($|r\rangle$) is the translational state of particles which passed through the left (right) slit (with the other slit closed), and $\uparrow_a$ is the rotated spin state of particles which passed through the right slit. In the position representation the translational states are $l(x) := \langle x|l\rangle \propto e^{ikx}$, $r(x) := \langle x|r\rangle \propto e^{-ikx}$.

The state vector (**??**) is that of an entangled state, yet it is not written in the form of a Schmidt decomposition. In contrast to the motional state "passage though the left slit" $|l\rangle$ and "passage through the right slit" $|r\rangle$, which are true alternatives, $\langle l|r\rangle = 0$, the spin state are not necessarily orthogonal.

(a) Compute the Schmidt decomposition of Eq. (**??**). Confirm

$$|\Psi\rangle = \sqrt{p}|\phi_+\rangle \otimes |\uparrow_c\rangle + \sqrt{1-p}|\phi_-\rangle \otimes |\downarrow_c\rangle \tag{2}$$

where $\vec{c} = (\vec{z}+\vec{a})/\|\vec{z}+\vec{a}\|$ is a spatial unit vector, $|\phi_\pm\rangle = (|r\rangle \pm |l\rangle)/\sqrt{2}$. What is p expressed in terms of $\vec{a}, \vec{z}$?

(b) The particles impinge on a detection screen (a CCD camera, say) which is not sensitive to the spin state. Compute the probability density I at point x on the detection screen. Confirm

$$I(x) \propto |l(x)|^2 + |r(x)|^2 + \beta l(x)^* r(x) + \beta^* r(x)^* l(x). \tag{3}$$

where

$$\beta = \langle \uparrow_z | \uparrow_a \rangle. \tag{4}$$

(c) The density I diplays an interference pattern, the modulation depth of which, called *fringe contrast*, depends sensitively on the spin-state overlap β. Using the definition of the fringe contrast

$$\gamma := \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} \tag{5}$$

please confirm

$$\gamma = |\beta|. \tag{6}$$

For maximally distinguishable spin states $\beta = 0$, the contrast is zero and the interference pattern turns into the distribution of classical point particles. For indistinguishable spin states, $\beta = 1$, contrast is maximal, i.e. the density distribution is "maximally quantum". Note that the reduction in contrast takes place even though, in our model, the particle's motion is not influenced by the magnetic field.

Yet spin measurement may reveal information about the slit each invidual particle took before reaching the screen, called *which-path-information*. The more which-path information we can extract, the lower the fringe contrast, and concomitantly, the more "classical" the density distribution.

(d) Use the rules of elementary quantum mechanics to confirm

$$\mathrm{Prob}(l,\uparrow_z) = \tfrac{1}{2}\,, \qquad \mathrm{Prob}(r,\uparrow_z) = \tfrac{1}{2}q\,, \tag{7}$$
$$\mathrm{Prob}(l,\downarrow_z) = 0\,, \qquad \mathrm{Prob}(r,\downarrow_z) = \tfrac{1}{2}\left(1-q\right)\,. \tag{8}$$

where $q = |\beta|^2$.

(e) Infer $\mathrm{Prob}(\uparrow_z) = \frac{1}{2}(1+q)$, $\mathrm{Prob}(\downarrow_z) = \frac{1}{2}(1-q)$, and the conditional probabilities

$$\mathrm{Prob}(l|\uparrow_z) = \frac{1}{1+q}\,, \qquad \mathrm{Prob}(r|\uparrow_z) = \frac{q}{1+q}\,, \tag{9}$$
$$\mathrm{Prop}(l|\downarrow_z) = 0\,, \qquad \mathrm{Prob}(r|\downarrow_z) = 1\,. \tag{10}$$

(f) The conditional probabilities, in turn, can be quantified in terms of conditional entropies. Cofirm that the residual uncertainty, which remains about the path given the particle is detected with spin up, is given by

$$H(\mathrm{path}|\uparrow_z) = \frac{1}{1+q}\log_2\left(1+q\right) + \frac{q}{1+q}\log_2\left(\frac{1+q}{q}\right)\,, \tag{11}$$

while $H(\mathrm{path}|\downarrow_z) = 0$, because $\downarrow_z$ can only be found for particles which took the right slit.

(g) The initial level of ignorance about the path is $H_{\mathrm{initial}} = 1bit$. The average level of ignorance which remains after a spin measurement is $H_{\mathrm{final}} = \mathrm{Prob}(\uparrow_z)H_\uparrow + \mathrm{Prob}(\downarrow_z)H_\downarrow$. The average information gain $I_{\mathrm{av}} = H_{\mathrm{initial}} - H_{\mathrm{final}}$. Compute I_{av} and summarize: Information gain is maximal if the spin states $\uparrow_z$, $\uparrow_a$ are orthogonal, $q = 0$. If the spin states are parallel, $q = 0$, information gain is zero – in this case spin measurement does not reveal any which-path-information.

Evidently, it is the mere presence of which-path information, and not the uncontrolled scattering of a photon, say, which affects the spatial density distribution. The more we can learn about the path, the more classical appears the distribution. The less we can learn about the path, the more quantum appears the distribution.

In our model, the which-path information and fringe contrast is intimately linked to the entanglement between the motional and spin degrees of freedom. The more entanglement, the better the which-path measurement. The better the which-path measuremnt, the less quantum the pattern. The less quantum the pattern, the more classical the distribution. Entanglement may well destroy that what is most important – the coherence. In the present case it destroys the coherence between the wave functions $l(x)$ and $r(x)$.

▷ **Aufgabe 2 (Landauer's Principle)** (8 scores)

The Landauer's principle states that the erasure of one bit of information inevitably generates heat $Q = k_{\mathrm{B}}T \ln 2$, where T is the temperature of the computing enviroment.

(a) Can you find a simple proof which is based on the elementary thermodynamics of a single particle ideal gas?

Now introduce Maxwell's Demon – a small measurement device which can first measure the bit value, and then – at no cost of energy and with no increase of the gas entropy – erases the bit value by putting the gas into a pre-defined "initial state".

(b) At first sight, Maxwell's Demon seems to contradict Landauer's Principle. Why?

(c) Yet the Maxwell's Demon in fact does not contradict the Landauer's Principle. Why not?

Hint: You may want to read the article by M.B. Plenio and V. Vitalli *The physics of forgetting: Landauers erasure principle and information theory*, Cont. Phys. **42**, 25–60 (2001).

Lösungen zu den Aufgaben