Theoretische Physik V
- Quantenmechanik II (WS 2018/2019) -

Übungsblatt 3

Ausgabe 06.11.18 – Abgabe 22.11.18 – Besprechung 23.11.18

▹ Aufgabe 1

Consider a two-level atom which was prepared at time $t = 0$ in its excited state $|e\rangle$. After sufficiently long a time, one will find, with high probability, the atom in its ground state $|g\rangle$: while waiting, the atom has spontaneously – i.e. without external intervention – emitted a photon, and thereby made a transition into its ground state. If you insist, you may describe this process within the framework of the Schrödinger equation, but in this case you would be forced to consider not only the dynamics of the atom, but the temporal evolution of a combined system "atom+electromagnetic field". Yet not being interested in the field degrees of freedom, but only in the atom, there may be a dynamical equation which describes the atomic evolution without being hazzled by the infinitely many degrees of freedom of the electromagnetic field.

The equation in question, the derivation of which was outlined in the lecture, reads

$$\frac{d}{dt}\hat{\varrho} = -\frac{i}{\hbar}\left[\hat{H}_{\text{eff}}\hat{\varrho} - \hat{\varrho}\hat{H}_{\text{eff}}^{\dagger}\right] + \gamma\hat{\sigma}\hat{\varrho}\hat{\sigma}^{\dagger}\,, \tag{1}$$

where $\hat{\sigma} = |g\rangle\langle e|$, and *effective Hamiltonian*

$$\hat{H}_{\text{eff}} = \hbar\left(\omega_0 - i\frac{\gamma}{2}\right)\hat{\sigma}^{\dagger}\hat{\sigma}\,. \tag{2}$$

Here, the parameters ω_0 and γ refer to the Bohr transition frequency and the rate of spontaneous emission (natural linewidth) or Einstein A-coefficient, respectively. The inverse $1/\gamma$ is also called natural life time (of the excited state $|e\rangle$).

(a) Show that the atomic state can be expanded

$$\hat{\varrho}(t) = \frac{1}{2}\left(\hat{\sigma}_0 + u(t)\hat{\sigma}_1 + v(t)\hat{\sigma}_2 + w(t)\hat{\sigma}_3\right) \tag{3}$$

with u, v, w real functions of time, $\hat{\sigma}_0$ the identity on the Hilbertspace of the two-level atom, and $\hat{\sigma}_i$, $i = 1, 2, 3$, the standard Pauli-Operators.[1] What ist the physical meaning of u, v and w?

(b) Derive and solve the dynamical equations for the functions u, v, w.

(c) The solution of the Liouville-von Neumann equation (1) provides a map on state space, $\hat{\varrho}_0 \mapsto \hat{\varrho}_t = e^{\mathcal{L}t}\hat{\varrho}_0$. In order to qualify for a possible physical process, this map must necessarily be (i) positive, and (ii) trace preserving. Show that this is indeed the case.

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[1] $\hat{\sigma}_1 = \hat{\sigma} + \hat{\sigma}^{\dagger}$, $\hat{\sigma}_2 = i(\sigma - \sigma^{\dagger})$, $\hat{\sigma}_3 = \hat{\sigma}^{\dagger}\hat{\sigma} - \hat{\sigma}\hat{\sigma}^{\dagger}$.