Einführung in die Quantenoptik I

Wintersemester 2018/19
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Übungsaufgaben Blatt 5 Hand out: 11 Dec 2018
Hand in: 19 Dec 2018

Aufgabe 5.1 – Radiation in the Universe, qualitatively (10 Punkte)

For the following discussion, you may use that a temperature T corresponds to a characteristic wavelength $\lambda_T = \hbar c / k_B T$ (Wien's displacement law), at least for massless particles. In thermal equilibrium, these particles appear with an energy density $u \sim k_B T / \lambda_T^3$.

(1) Compute the average photon number at room temperature for visible light and for the frequency band used by your mobile phone.

(2) Find out precise values of the photosphere temperature of the Sun and calculate the corresponding thermal wavelength $\lambda_\odot$. In which frequency band does this fall?

(3) When the Universe was about three minutes old, it was in thermal equilibrium so that the thermal energy $k_B T$ was comparable to the rest mass of the electron. Estimate this temperature and the corresponding energy density. Which were the most abundant particles in this epoch? Consider protons and neutrons in equilibrium at this temperature and speculate why the ratio $\exp[(m_n - m_p)c^2 / k_B T]$ gives an idea of the ratio between proton and neutron densities. How big is the number?

(4) In the energy of the quantized radiation field, we often 'neglect' the contribution of the zero-point energy of each field mode, given by $\frac{1}{2}\hbar\omega_k$ per mode. This 'small term' leads, in the vacuum state of the electromagnetic field, to an energy density u_{vac} that depends on the cutoff wavenumber k_c as follows

$$u_{\text{vac}} = C \hbar c k_c^4 \tag{5.1}$$

where C is a numerical constant 'of order unity'. Check that Eq.(5.1) is consistent with the Stefan-Boltzmann law for the energy density in the blackbody radiation field, provided a "vacuum temperature" $T_{\text{vac}} = \hbar c k_c$ is introduced.

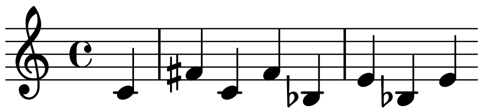
(5) Fix the cutoff for the vacuum energy at the energy scale $E_{\text{GUT}} = \hbar c k_c$ for the 'unification' of the fundamental interactions (electroweak and strong, except gravity, look up the keyword 'grand unified theory') and make an estimate for the corresponding vacuum energy density. Express this number in atomic mass units (times c^2) per

cubic meter. [Hoax: "this solves the energy crisis."] Look up the length scale ℓ_{Planck} for the unification of gravity, relativity, and quantum theory and compare the vacuum energy corresponding to $k_c = 1/\ell_{\text{Planck}}$ to the observed mass density in the Universe and the density of 'dark energy' (responsible for the accelerated expansion of the Universe, Nobel prize in physics 2011). [Jargon: "wrongest formula in all physics."]

For an informal exposition of this so-called "cosmological constant problem", see Adler, Casey, and Jacob [*Am. J. Phys.* **63** (1995) 620]. For a recent attempt to solve it, see Wang, Zhu, and Unruh [*Phys. Rev. D* **95** (2017) 103504 = arxiv:1703.00543].

Aufgabe 5.2 – Maria mit dem Kind (4 Punkte)

Sie stehen an der Straße, und Maria fährt auf dem Weg zur Geburtshilfe vorbei:



Wie schnell fährt die Ambulanz? Welches Musical wird (wohl unbewusst) zitiert?

Aufgabe 5.3 – Field commutators (6 Punkte)

In the lecture, we have seen the commutator between the field operators $\mathbf{E}(x)$ and $\mathbf{H}(x)$:

$$[E_j(\mathbf{x}, t),\ H_k(\mathbf{x}', t)] = (\text{const.})\mathrm{i}\hbar\epsilon_{jkl}\frac{\partial}{\partial x_l}\delta(\mathbf{x} - \mathbf{x}') \tag{5.2}$$

where the constant depends on the system of units. All other commutators vanish. All fields in this problem have to be understood as operators.

(1) Fix the constant in Eq.(5.2) by requiring that the Heisenberg equation of motion for the magnetic field,

$$\partial_t \mathbf{H} = \frac{\mathrm{i}}{\hbar}[H, \mathbf{H}] \tag{5.3}$$

reproduces Faraday's induction equation. Here H is the space integral of the electromagnetic energy density.

(2) Consider the amplitude operators for smooth mode functions $\mathbf{f}$ and $\mathbf{h}$:

$$\mathcal{E} = \int \mathrm{d}^3x\, \mathbf{f}(\mathbf{x}) \cdot \mathbf{E}(\mathbf{x}), \qquad \mathcal{H} = \int \mathrm{d}^3x\, \mathbf{h}(\mathbf{x}) \cdot \mathbf{H}(\mathbf{x}) \tag{5.4}$$

and assume that the modes are related by $\omega\varepsilon_0 \mathbf{f} = \nabla \times \mathbf{h}$. Work out the commutator $[\mathcal{E}, \mathcal{H}]$ and write a few sentences about the consequences: which observables cannot be measured simultaneously? Derive the uncertainty relation between the variances $(\Delta\mathcal{E})^2$, $(\Delta\mathcal{H})^2$.