

Chapter 3

Photodetection and the Optical Master Equation

3.1 Lectures 22/29 Jan 19: Glauber's model

A description based on a general master equation can be found in Sec.3.3.

3.2 Photodetector theory à la Glauber

We now have to look in more detail at how exactly the quantum state of the electromagnetic field can be measured. The devices that measure light are called “photodetectors”. We shall sketch here the theory that R. Glauber developed to describe these detectors. Light detection has some particular properties: for example, it is practically impossible to resolve the oscillations of the electric field at an optical period of about 10^{-15} s — detectors are simply “too slow” for that. They have a “response time” that is typically in the 10^{-9} s range (“fast” photodetector). In practice (and also in the classical theory), one therefore has only access to time-averaged quantities, the average being taken over at least one optical period.

A good “quantum” model for a light detector is a two-level atom initially in its ground state that gets excited when it is illuminated by light. If we can measure the population of its excited state, we have a signal which is related to the light field. This is the starting point of Glauber's theory.

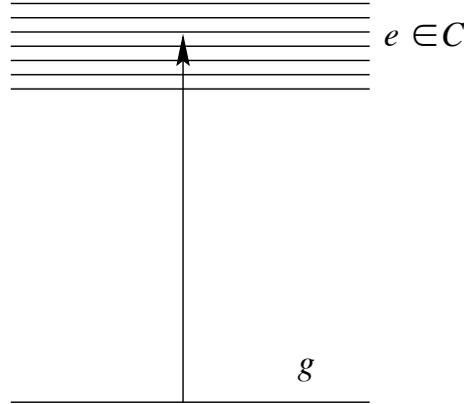


Figure 3.1: Two-level model for a photodetector.

Detector model. The details are the following: an efficient detection is possible when the two-level system is ionized by the light because the ejected electron can efficiently be detected (this is used in photomultipliers). This means, however, that our excited state is actually a continuum (the electron moves freely), and we cannot use a simple-minded two-state model. So let us assume the level scheme shown in figure 3.3 with a ground state $|g\rangle$ and a continuum C of excited states $|e\rangle$.

We are interested in the probability $p_C(t)$ that this detector goes into any of the excited states $|e\rangle$ (“detection probability”). In terms of the probability amplitudes $c_{e,g}(t)$, the detection probability is given by

$$p_C(t) = \sum_{e \in C} |c_e(t)|^2. \quad (3.1)$$

We start at $t = 0$ with the detector in the ground state. To compute the $c_e(t)$, nothing better can we do than perturbation theory, as we did in chapter 1. The result for the state $|\tilde{\psi}(t)\rangle$ in the interaction picture is

$$|\tilde{\psi}(t)\rangle \approx |\tilde{\psi}(0)\rangle - \frac{i}{\hbar} \int_0^t dt' V(t') |\tilde{\psi}(0)\rangle \quad (3.2)$$

Projecting this on the final state with one electron excited, we get

$$c_e(t) = -\frac{i}{\hbar} \int_0^t dt' \langle e; f | V(t') | g; i \rangle e^{i\omega_{eg}t'} \quad (3.3)$$

where $|i\rangle$ and $|f\rangle$ are the initial and final field states. The interaction potential is taken in the electric dipole coupling

$$V(t) = -\hat{\mathbf{d}} \cdot \mathbf{E}(t) = -\hat{\mathbf{d}} \cdot \mathbf{E}^+(t) - \hat{\mathbf{d}} \cdot \mathbf{E}^-(t) \quad (3.4)$$

where $\mathbf{E}(t)$ is the electric field operator at the location of the detector, and $\mathbf{E}^\pm(t)$ its positive and negative frequency part: in the quantized description, the positive frequency part $\mathbf{E}^+(t)$ only contains annihilation operators $a_{\mathbf{k}}(t) \propto e^{-i\omega_{\mathbf{k}}t}$, and the negative frequency part only creation operators $a_{\mathbf{k}}^\dagger(t) \propto e^{i\omega_{\mathbf{k}}t}$.

Detection probability. Also in the quantum theory, we can make the resonance approximation (or equivalently, the r.w.a.) and keep in the time-integral (3.3) only those terms that lead to resonant denominators. Since the transition frequency ω_{eg} is positive, this amounts to keeping only the positive frequency part of the electric field operator. We thus have

$$c_e(t) = \frac{i}{\hbar} \mathbf{d} \cdot \int_0^t dt' \langle f | \mathbf{E}^+(t') | i \rangle e^{i\omega_{eg}t'} \quad (3.5)$$

where we have used the abbreviation $\mathbf{d} = \langle e | \hat{\mathbf{d}} | g \rangle$. For the probability $p_C(t)$, we get an expression involving $\langle i | E_j^-(t'') | f \rangle \langle f | E_i^+(t') | i \rangle$. At this point, we can perform the sum over the final field states $|f\rangle$ (these are not resolved, so we add the corresponding probabilities). We thus get

$$p_C(t) = \frac{1}{\hbar^2} \sum_{e \in C} d^{j*} d^i \int_0^t dt' dt'' \langle i | E_j^-(t'') E_i^+(t') | i \rangle e^{i\omega_{eg}(t'-t'')} \quad (3.6)$$

An explicit calculation of the sum over the continuum E is only possible with a specific model. Let us assume here the following form

$$\sum_{e \in C} d^{j*} d^i e^{i\omega_e \tau} = D \delta^{ij} \delta(\tau). \quad (3.7)$$

This form is plausible when the width $\Delta\omega_e$ of the continuum is sufficiently broad: for τ of the order of the time resolution of the detector, the phase factors $e^{i\omega_e \tau}$ then rapidly average out. Note that this behaviour is a consequence of the fact that “photodetectors are slow”. The Kronecker δ^{ij} for the field components is plausible when the detector does not discriminate the field polarization. This is a simplifying assumption that we relax later to describe polarization-sensitive detectors.

Putting (3.7) into the detection probability (3.6), we get the basic result

$$p_C(t) = \eta \int_0^t dt' \langle \mathbf{E}^-(t') \cdot \mathbf{E}^+(t') \rangle, \quad \eta = \frac{D}{\hbar^2}, \quad (3.8)$$

where we see that a photodetector is sensitive to a “normally ordered” product of electric field operators (annihilation operators to the right and act first). The average $\langle \dots \rangle$ is an average over the field state — which can be a quantum average (in a pure quantum state $|i\rangle$) or a combined quantum and classical average. If the field is described by the density operator $\hat{\rho}$, then we obviously have

$$\langle \mathbf{E}^-(t') \cdot \mathbf{E}^+(t') \rangle = \text{tr} \left(\hat{\rho} \mathbf{E}^-(t') \cdot \mathbf{E}^+(t') \right).$$

The time-dependence of the operators reminds us that we deal with Heisenberg operators – the density matrix thus specifies the “initial” state (that does not evolve in time).

Photocurrent. The time derivative of the detection probability gives the production rate of photoelectrons, or (up to a factor e) the photocurrent $j(\mathbf{r}, t)$:

$$\begin{aligned} j(\mathbf{r}, t) &= \eta \langle I(\mathbf{r}, t) \rangle, \\ I(\mathbf{r}, t) &= : \mathbf{E}^-(\mathbf{r}, t) \cdot \mathbf{E}^+(\mathbf{r}, t) : \end{aligned} \quad (3.9)$$

The notation $: \dots :$ means “normal order” (creation operators to the left, annihilation operators to the right and act first). Up to this operator ordering prescription, we have found an expression that resembles the classical time-averaged Poynting vector (recall that it is also proportional to the squared complex field amplitude). Note that the average here is more involved than in the classical theory, since it also has quantum aspects.¹ The proportionality factor η depends on the detector, and theoretical results are often normalized to make it disappear.

If the field is a single plane wave mode, we see that the photocurrent is proportional to the photon number operator:

$$j(t) = \eta_1 \langle a^\dagger(t) a(t) \rangle \quad (3.10)$$

with $\eta_1 = \eta E_{1\text{ph}}^2$. In this context, the factor η_1 is often interpreted as “detection efficiency”, i.e., the probability that a photon is detected.

¹A more precise theory is possible at the classical level (“optical coherence theory”): there, the light field is treated as a stochastic field with fluctuations, since it often originates from many uncorrelated sources (classical representation of individual atoms). In the theory, an extra average over these fluctuations then enters. Experimentally, this average enters because detectors are “slow” and cannot resolve these fluctuations.

Polarization filtering

Imagine a polarization filter in front of the photodetector. In classical terms, this means that only a component $\mathbf{u}^* \cdot \mathbf{E}$ is transmitted to the detector.² In the quantum theory, we can describe this by expanding the field $\mathbf{E}(\mathbf{r}, t)$ into modes polarized parallel and perpendicular to \mathbf{u} . Only the mode with parallel polarization has a mode function that reaches the detector (recall that the calculation of mode functions is a classical problem, independent of the field quantization). So our photocurrent becomes

$$j_{\mathbf{u}}(\mathbf{r}, t) = \langle \mathbf{u} \cdot \mathbf{E}^-(\mathbf{r}, t) \mathbf{u}^* \cdot \mathbf{E}^+(\mathbf{r}, t) \rangle$$

Frequency spectrum

What happens with filtering in frequency space? We can in fact use a very similar reasoning: a classical mode at frequency ω will be transmitted to the detector with an efficiency given by the transmission coefficient of the filter. This quantity can be calculated for a cavity, e.g., and shows peaks (maximum transmission) at specific resonance frequencies. Let us focus on one of these frequencies, say ω_c , and suppose that the filter has a very narrow resonance. The transmitted field is then proportional to the time-Fourier transformed electric field at the resonance frequency ω_c :

$$\mathbf{E}_{\text{tr}}^+(t) \propto \tilde{\mathbf{E}}^+(\omega_c) e^{-i\omega_c t}$$

The corresponding photocurrent is then given by

$$\begin{aligned} j_{\omega_c}(t) &\propto \langle \tilde{\mathbf{E}}^-(\omega_c) \cdot \tilde{\mathbf{E}}^+(\omega_c) \rangle \\ &= \int dt_1 dt_2 e^{-i\omega_c(t_1-t_2)} \langle \mathbf{E}^-(t_1) \cdot \mathbf{E}^+(t_2) \rangle \end{aligned} \quad (3.11)$$

If we assume that the field is stationary in time, we can shift the time arguments in the product of field averages and write

$$j_{\omega_c}(t) \propto T \int d\tau e^{i\omega_c \tau} \langle \mathbf{E}^-(0) \cdot \mathbf{E}^+(\tau) \rangle \quad (3.12)$$

where T is the measurement time (at least equal to the time needed to build up the field in the cavity). This equation can be generalized to any detector (“spectrometer”) that measures a frequency spectrum. Finally, if the field changes

²The complex conjugate \mathbf{u}^* is applicable when a filter for circular polarization is used.

slowly (on the timescale of the measurement time T), we can write

$$j_{\omega_c}(t) \propto T \int d\tau e^{i\omega_c\tau} \langle \mathbf{E}^-(t - \tau/2) \cdot \mathbf{E}^+(t + \tau/2) \rangle.$$

This quantity is called the ‘Wigner transform’ of $\langle \mathbf{E}^-(t) \cdot \mathbf{E}^+(t') \rangle$, a function that combines information in real time and in frequency space.

A result similar to Eq.(3.12) appears when we take into account the excited state continuum in Fig.3.3 is not ‘infinitely broad’. A simple generalization of Eq.(3.7) is the following

$$\sum_{e \in C} d^{j*} d^i e^{i\omega_e\tau} = \frac{D\gamma}{2} \delta^{ij} e^{i\omega_c\tau} e^{-\gamma|\tau|}. \quad (3.13)$$

where γ is proportional to the spectral width of the continuum states and ω_c their ‘center frequency’. When this is put into Eq.(3.6) for the ‘click’ probability $p_C(t)$, we find

$$p_C(t) = \frac{\eta\gamma}{2} \int_0^t dt' dt'' e^{i\omega_c(t'-t'')} e^{-\gamma|t'-t''|} \langle \mathbf{E}^-(t') \cdot \mathbf{E}^+(t'') \rangle, \quad \eta = \frac{D}{\hbar^2}, \quad (3.14)$$

(See also Eq.(3.55) for an approach based on the master equation for a two-level system.) The time derivative gives the ‘rate of clicks’ or the photodetector signal. If the measurement time t is much longer than the ‘detector response time’ $1/\gamma$, then this derivative has a similar form as Eq.(3.12) except for the factor $e^{-\gamma|t'-t''|}$ that naturally cuts off the integral. (In many cases, the correlation function $\langle \mathbf{E}^-(t') \cdot \mathbf{E}^+(t'') \rangle$ will decay to zero faster.)

3.3 Master equation formulation

Idea

Many systems in quantum optics are “open” and follow equations of motion that go beyond the Schrödinger equation. This applies to systems for which the Hamiltonian is not completely known or systems in contact with an “environment”. The equations of motion can be formulated as so-called master equations (similar to the Schrödinger picture, dynamics for the system density operator) or as so-called Langevin equations (similar to the Heisenberg picture, dynamics of the system operators). These equations are the quantum analogue of kinetic

theories that describe the dynamics of a system including different dissipative effects. They provide, among others, the approach of the system towards thermal equilibrium, but also non-equilibrium states that are driven by external forces, the typical example being a laser field.

In quantum optics, the “system” can be an atom, a collection of atoms, or a field mode in a cavity. The system is “open” because it interacts with the “environment” (rest of the world), that is modelled as a continuum of quantized field modes. An atom decays irreversibly by emitting a photon into previously empty vacuum modes. This phenomenon of “spontaneous emission” does not allow for a description in terms of a Hamiltonian. In fact, the state of the atom does not remain pure, and entropy increases because the photon can be emitted into any direction of space. A cavity mode decays because photons escape through the cavity mirrors, effectively becoming excitations of the “modes outside the cavity”. Alternatively, the field energy can be absorbed by the material making up the mirrors.

We focus first on the derivation of a master equation for a two-level atom. We shall find from the general theory two results: (1) the rate of spontaneous emission of an electronically excited state and (2) Glauber’s formula for the signal of a photodetectors in terms of normally ordered correlation functions of the electric field operator.

Time scales

The derivation builds on the following hierarchy of time scales:

$$\frac{2\pi}{\omega_A} \leq \tau_c \ll \frac{2\pi}{\Omega} \sim \frac{1}{\gamma} \quad (3.15)$$

where the shortest time scale is the optical period of the light field that is near-resonant with the atomic transition frequency ω_A . This is typically smaller than a few fs (femtoseconds) (in the visible range).

The “correlation time” τ_c of the electromagnetic field describes qualitatively the time interval over which the quantum fluctuations of the electromagnetic field (that couple significantly to the atom) are “smooth”. Two electric field measurements are not significantly “similar” if they are taken at intervals larger than τ_c . We give an estimate below, it turns out to be a few times larger than the optical period.

The third time scale is the Rabi period that scales with the inverse of the Rabi frequency of a laser. This depends of course on the laser power (and the atomic transition dipole), but typical values are in the 1–10 ns (nanoseconds) range. Note that this is much longer than the optical period and the field correlation time.

The last time scale is the lifetime of the excited state of the atom, inversely proportional to the spontaneous decay rate γ . This is typically in the 1–10 ns range as well, depending on the atom.

We thus have a situation similar to “Brownian motion”: the atom is “shaken around” by the vacuum field fluctuations that vary much faster than the typical dynamics for the atomic state (Rabi oscillations, spontaneous decay). The master equation takes advantage of this separation of time scales to find an equation of motion that can be applied on “slow time scales”. As a result, one gets a single equation that describes both, the “Hamiltonian” effects of a laser (Rabi oscillations) and the “dissipative / friction” effects due to vacuum fluctuations.

Hamiltonian

Two-level atom, field, and electric dipole interaction:

$$H = \frac{\hbar\omega_A}{2}\sigma_3 + \sum_k \hbar\omega_k a_k^\dagger a_k + \sum_k \hbar \left(g_k^* a_k^\dagger \sigma + g_k \sigma^\dagger a_k \right) \quad (3.16)$$

neglect the zero-point energy of the field. Coupling constant $\hbar g_k = -\mathbf{d}_{eg} \cdot \mathbf{f}_k(\mathbf{x}_A)(\hbar\omega_k/2\varepsilon_0)^{1/2}$ with the mode function $\mathbf{f}_k(\mathbf{x})$ evaluated at the position of the atom. Sometimes called ‘vacuum Rabi frequency’.

In the following, we also use the (re-scaled) electric field operator

$$E(t) = \sum_k g_k a_k(t) + \text{h.c.} \quad (3.17)$$

Field correlation time

To get an estimate of the field correlation time, we compute the two-time correlation (or coherence) function

$$C(\tau) = \langle E(t + \tau)E(t) \rangle \quad (3.18)$$

If we use the coupling constants g_k to write the mode expansion of the quantized field (this is equivalent to re-scaling the electric field so that it has units of frequency), we get

$$\begin{aligned}\langle E(t + \tau)E(t) \rangle &= \sum_k |g_k|^2 \langle a_k(t + \tau)a_k^\dagger(t) \rangle \\ &= \sum_k |g_k|^2 e^{-i\omega_k \tau}\end{aligned}\quad (3.19)$$

An explicit calculation using the mode amplitudes in free space leads to an integral of the form

$$C(\tau) \sim \int_0^\infty d\omega \omega^3 e^{-i\omega \tau} \quad (3.20)$$

which looks like a third derivative of a δ -function in τ . Well, not exactly since the integrations starts at $\omega = 0$. Generalizing to finite temperature, the Fourier transform of the correlation function (3.20) becomes

$$C(\omega) \sim \omega^3 \bar{n}(\omega) = \frac{\omega^3}{\exp(\hbar\omega/k_B T) - 1} \quad (3.21)$$

The sign convention of the Fourier transform is chosen here such that $\omega > 0$ corresponds to photon numbers $a^\dagger a$ ('normal order'), while at $\omega < 0$, the anti-normal order aa^\dagger is picked. See Fig.3.2: the spectrum is proportional to $-\omega^3$ for negative frequencies, $-\omega \gg k_B T/\hbar$. Near zero frequency, it has a quadratic behaviour, at positive frequencies, a maximum near $\omega = 2k_B T/\hbar$ (Wien displacement law) and an exponential decay for $\omega \gg k_B T/\hbar$.

Actually, we forgot one thing in this estimate: the Hamiltonian we start with is only valid for field modes k near-resonant with the atomic transition, $\omega_k \sim \omega_A$. It thus seems reasonable to restrict the frequency integral in Eq.(3.20) to an interval around ω_A , with a width $\Delta\omega$ that is typically smaller than ω_A .

Now, from the properties of the Fourier transformation, we know that the correlation function $C(\tau)$ given by the integral (3.20) has a "width in time" given by $\Delta\tau \Delta\omega \sim 1$. Since this width is precisely the definition of the correlation time, we have

$$\frac{1}{\omega_A} \leq \frac{1}{\Delta\omega} \sim \tau_c. \quad (3.22)$$

Typical value: a few 10 fs.

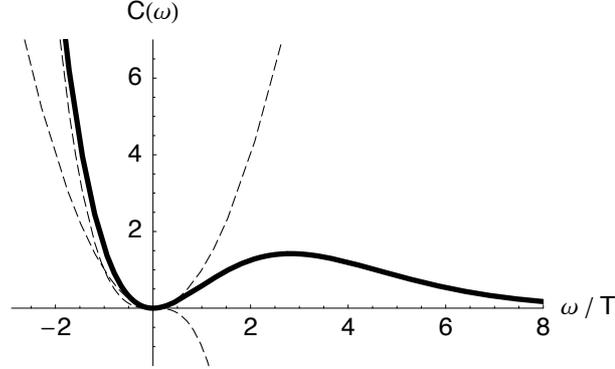


Figure 3.2: Spectrum $C(\omega)$, Eq.(3.21), of the radiation field (normal order) at finite temperature. The frequency is plotted in units of $k_B T/\hbar$. The dashed lines are ω^2 and $-\omega^3$.

3.3.1 Evolution of the atomic operators

We shall work in the Heisenberg picture. Note that operators describing different degrees of freedom (field, atom) commute at equal times.

Atomic dipole

“eliminate” the field operators by solving their equation of motion:

$$\frac{da_k}{dt} = \frac{i}{\hbar} [H, a_k] = -i\omega_k a_k - ig_k^* \sigma \quad (3.23)$$

$$a_k(t) = a_k(0) e^{-i\omega_k t} - ig_k^* \int_0^t dt' \sigma(t') e^{-i\omega_k(t-t')} \quad (3.24)$$

the “particular solution” contains the “past” of the atomic dipole operator $\sigma(t')$. Insert this into the equation for σ . Let’s look at this equation first:

$$\begin{aligned} \frac{d\sigma}{dt} &= \frac{i}{\hbar} [H, \sigma] = -i\omega_A \sigma + i \sum_k g_k [\sigma^\dagger, \sigma] a_k \\ &= -i\omega_A \sigma + i \sum_k g_k \sigma_3 a_k \end{aligned} \quad (3.25)$$

We see here that the two-level atom leads to nonlinear equations of motion: the operator product $\sigma_3 a_k$ appears. For the moment, these operators commute (at equal times). But we now want to insert the solution (3.24) for $a_k(t)$, and

the two terms that appear here do not commute separately with σ_3 . For this reason, we take now a specific operator order (so-called “normal order”) where the annihilation operators (a_k or σ) act first. This is the order already used in Eq.(3.25), and we thus get

$$\frac{d\sigma}{dt} = -i\omega_A\sigma + i\sigma_3\mathcal{E}_0(t) + \int_0^t dt' \sum_k |g_k|^2 e^{-i\omega_k(t-t')} \sigma_3(t)\sigma(t') \quad (3.26)$$

where we used the abbreviation

$$\mathcal{E}_0(t) = \sum_k g_k a_k(0) e^{-i\omega_k t} \quad (3.27)$$

for the freely evolving electric field operator (positive frequency component only, and re-scaled by the atomic transition dipole). In the integral over t' , we see that the correlation function $C(t-t')$ appears, and using $\tau = t-t'$ as integration variable, we have

$$\frac{d\sigma}{dt} = -i\omega_A\sigma + i\sigma_3\mathcal{E}_0(t) + \int_0^t d\tau C(\tau)\sigma_3(t)\sigma(t-\tau) \quad (3.28)$$

Now comes the *key observation*: under the time integral occur two very different functions. The correlation function $C(\tau)$ is very narrow in τ . The atomic operator $\sigma(t-\tau)$ contains a “fast free evolution” (generated by the first term in its equation of motion), but once this is factored out, we anticipate that its evolution is “slow”:

$$\sigma(t-\tau) = e^{-i\omega_A(t-\tau)}\tilde{\sigma}(t-\tau) \approx e^{-i\omega_A(t-\tau)}\tilde{\sigma}(t) = e^{+i\omega_A\tau}\sigma(t) \quad (3.29)$$

Here, we introduced temporarily the dipole operator $\tilde{\sigma}(t')$ (in an interaction picture).

The main benefit of this approximation is that only atomic operators at time t appear in the equation of motion. The two-level commutation rules³ give $\sigma_3(t)\sigma(t) = -\sigma(t)$, and we end up with

$$\frac{d\sigma}{dt} = -i\omega_A\sigma + i\sigma_3\mathcal{E}_0(t) - \sigma(t) \int_0^t d\tau C(\tau) e^{i\omega_A\tau} \quad (3.30)$$

³A pedantic remark: operator products evolve as products under the Heisenberg equations of motion. This is because taking the commutator with a product is compatible with the product rule of (time) derivatives.

The upper integration limit t is actually the difference between our initial time and the actual time where the equation of motion is computed. We now let this time difference be much larger than the correlation time τ_c . This is consistent with the assumption that the atomic dynamics is slow on the scale of the field's correlation time. Then the integrand is effectively zero at the upper limit, and we get a constant complex number

$$\gamma + i\delta\omega_A = \int_0^{t \gg \tau_c} d\tau C(\tau) e^{i\omega_A \tau} = \frac{S(\omega_A)}{2} + i\mathcal{P} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{S(\omega)}{\omega - \omega_A} \quad (3.31)$$

where $S(\omega)$ is the Fourier transform of the correlation function $C(\tau)$ and \mathcal{P} means the principal part of the integral.

3.3.2 Spontaneous decay rate and Lamb shift

Explicit calculation: spectrum of vacuum fluctuations

$$S(\omega) = 2\pi \sum_k |g_k|^2 \delta(\omega_k - \omega) \quad (3.32)$$

Take a plane wave expansion and sum over the polarization vectors $\mathbf{u}_{\mathbf{k}\lambda}$ in the coupling constants g_k

$$\sum_\lambda |\mathbf{d}_{\text{eg}} \cdot \mathbf{u}_{\mathbf{k}\lambda}|^2 = |\mathbf{d}_{\text{eg}}|^2 - |\mathbf{d}_{\text{eg}} \cdot \hat{\mathbf{k}}|^2 \quad (3.33)$$

where $\hat{\mathbf{k}}$ is the unit vector along \mathbf{k} . This formula arises because the $\mathbf{u}_{\mathbf{k}\lambda}$ are perpendicular to \mathbf{k} . Integration over the angles of \mathbf{k} gives

$$\int d\Omega(\hat{\mathbf{k}}) (|\mathbf{d}_{\text{eg}}|^2 - |\mathbf{d}_{\text{eg}} \cdot \hat{\mathbf{k}}|^2) = 4\pi |\mathbf{d}_{\text{eg}}|^2 - \frac{4\pi}{3} |\mathbf{d}_{\text{eg}}|^2 = \frac{8\pi}{3} |\mathbf{d}_{\text{eg}}|^2 \quad (3.34)$$

The integral over the length of \mathbf{k} is trivial because of the δ -function in Eq.(3.32), its length is fixed to $|\mathbf{k}| = \omega/c$. Putting everything together, we get

$$S(\omega) = \frac{2\pi}{3\hbar^2} |\mathbf{d}_{\text{ge}}|^2 \frac{\hbar\omega}{2\varepsilon_0} \frac{8\pi\omega^2}{(2\pi c)^3} \quad (3.35)$$

where the last factor is the density of field modes per $d\omega$ and volume. We can thus say that the spontaneously decaying atom is a “detector” for vacuum field fluctuations. The decay rate γ

$$\gamma = \frac{1}{2} S(\omega_A) = \frac{|\mathbf{d}_{\text{ge}}|^2 (\omega_A/c)^3}{6\pi\hbar\varepsilon_0} \quad (3.36)$$

is also called the “natural linewidth” of the atomic transition $|g\rangle \leftrightarrow |e\rangle$ because it gives the width in frequency of the spontaneous emission spectrum. In order of magnitude, with $|\mathbf{d}_{ge}| \sim ea_0$ (electron charge \times atom size)

$$\frac{\gamma}{\omega_A} \sim \alpha_{\text{fs}}(a_0/\lambda_A)^2 \sim \alpha_{\text{fs}}^3 \quad (3.37)$$

with fine structure constant $\alpha_{\text{fs}} = e^2/4\pi\epsilon_0\hbar c \approx 1/137.04$ and wavelength λ_A of resonant transition. Hence, indeed decay is “slow” on the scale of the optical period.

Frequency shift $\delta\omega_A$ is related to asymmetry of vacuum spectrum around the transition frequency. Interpretation from second order perturbation theory: modes slightly below the atomic resonance, $\omega_k \leq \omega_A$ tend to push the level $|e\rangle$ upwards. Modes above resonance: push downwards. Null effect if spectrum is flat.

Actual calculation requires UV cutoff ω_{uv} and interaction Hamiltonian beyond the resonance (rotating wave) approximation (Hans Bethe \sim 1948, calculation of the Lamb shift). Order of magnitude:

$$\delta\omega_A \approx \gamma \log(\omega_{\text{uv}}/\omega_A) \quad (3.38)$$

and a consistent cutoff is the electron rest mass, $\omega_{\text{uv}} \sim m_e c^2/\hbar$. Hence comparable to the natural linewidth. Note: relativistic theory required, where wavelengths up to electron Compton wavelength $\hbar/m_e c$ are involved. This goes beyond the “long wavelength approximation” because $\hbar/m_e c \sim \alpha_{\text{fs}} a_0 \ll a_0$.

Atomic populations

Actually, we did not yet show that γ is the rate of decay for the excited state. For this, we need the equation of motion for the occupations of the two energy levels. This is described by the atomic operator σ_3 , also called the “inversion” because $\langle \sigma_3 \rangle > 0$ when the excited state is more occupied than the ground state.

Heisenberg equation of motion

$$\begin{aligned} \frac{d\sigma_3}{dt} &= +2i \sum_k [g_k^* a_k^\dagger \sigma - g_k \sigma^\dagger a_k] \\ &= 2i [\mathcal{E}_0^\dagger(t) \sigma - \sigma^\dagger \mathcal{E}_0(t)] \\ &\quad - 2 \int_0^t dt' [C^*(t-t') \sigma^\dagger(t') \sigma(t) + C(t-t') \sigma^\dagger(t') \sigma(t)] \end{aligned} \quad (3.39)$$

where we inserted the formal solution for $a_k(t)$ and brought the operator products in normal order. We apply to the t' -integral the same prescription as before and get

$$\begin{aligned} & \int_0^t dt' \left[C^*(t-t')\sigma^\dagger(t')\sigma(t) + C(t-t')\sigma^\dagger(t')\sigma(t) \right] \\ & \approx (\gamma - i\delta\omega_A)\sigma^\dagger(t)\sigma(t) + (\gamma + i\delta\omega_A)\sigma^\dagger(t)\sigma(t) = \gamma(\sigma_3(t) + \mathbb{1}) \end{aligned} \quad (3.40)$$

where the frequency shift drops out. By construction, the operator $\sigma_3(t) + \mathbb{1}$ gives the occupation of the excited state. From Eqs.(3.39, 3.40), we can thus read off the decay rate 2γ for the excited state population, while the ground state remains stable.

Finally, the equation for the inversion operator becomes

$$\frac{d\sigma_3}{dt} = -2\gamma(\sigma_3(t) + \mathbb{1}) + 2i \left[\mathcal{E}_0^\dagger(t)\sigma - \sigma^\dagger\mathcal{E}_0(t) \right] \quad (3.41)$$

Combined with the equation of motion for the atomic dipole operator,

$$\frac{d\sigma}{dt} = -(\gamma + i\omega_A)\sigma + i\sigma_3\mathcal{E}_0(t) \quad (3.42)$$

we have thus found a result that looks similar to the “optical Bloch equations” (more details in the next section).

In Eq.(3.42), we have made the replacement $\omega_A + \delta\omega_A \mapsto \omega_A$ for the atomic frequency. This is called “renormalization”: we combine the shift induced by the coupling to the vacuum field with the “naked” transition frequency into the frequency that can be physically observed. Recall that in reality, we can never “switch off” the coupling to the vacuum. Hence the “naked” two-level atom that we started with is actually a theoretical artefact.

3.3.3 Application 1: Bloch equations

Note that we did not specify yet the state of the light field: it is in fact encoded in the operator $\mathcal{E}_0(t)$ that depends on the initial field operators $a_k(0)$. Two examples will be studied now, the first one being an atom driven by a laser field. We have argued that to a good approximation, we can assume that the light field is in a coherent state $|\alpha_L\rangle$. We assume that at $t = 0$, the total system is in the product state $|\psi(0), \alpha_L\rangle$ and take the expectation value of the Bloch equation. This gives

the dynamics of the Bloch vector as follows (optical Bloch equations in the proper sense)

$$\frac{ds}{dt} = -(\gamma + i\omega_A)s + (i/2)s_3\Omega e^{-i\omega_L t} \quad (3.43)$$

$$\frac{ds_3}{dt} = -2\gamma(s_3(t) + 1) + i \left[\Omega^*(t)e^{i\omega_L t}s - s^*\Omega e^{-i\omega_L t} \right] \quad (3.44)$$

where $\Omega/2 = \langle \alpha_L | \mathcal{E}_0(0) | \alpha_L \rangle$ is the (complex) Rabi frequency and ω_L the frequency of the laser mode.

These equations have time-dependent coefficients, but this can be removed by making a transformation into a “rotating frame”. We make the replacement

$$s(t) \mapsto s(t)e^{-i\omega_L t} \quad (3.45)$$

where the “new” $s(t)$ satisfies Bloch equations with time-independent coefficients

$$\begin{aligned} \frac{ds}{dt} &= -(\gamma - i\delta)s + (i/2)s_3\Omega \\ \frac{ds_3}{dt} &= -2\gamma(s_3(t) + 1) + i[\Omega^*s - s^*\Omega] \end{aligned} \quad (3.46)$$

where $\delta = \omega_L - \omega_A$ is the “laser detuning”. Note that the symbols are not the same throughout the books: the Rabi frequency Ω or the decay rate γ can differ by a factor of 2 (or -2), the detuning can have the opposite sign.

The Bloch equations are a “workhorse” of atomic physics and quantum optics. They are used to compute light absorption, excitation spectra, population transfer, radiation forces on atoms etc. In the exercises, you compute the stationary state of the Bloch equations (attention with the different sign for δ : this one should be correct)

$$s_{ss} = \frac{-i(\Omega/2)(\gamma + i\delta)}{\gamma^2 + \delta^2 + \Omega^2/2} \quad (3.47)$$

$$s_{3,ss} = \frac{-(\gamma^2 + \delta^2)}{\gamma^2 + \delta^2 + \Omega^2/2} \quad (3.48)$$

Discussion: average atomic dipole operator (induced by laser field), average inversion. Line broadening.

Exercise: total excitation N , does not commute when laser included. Expectation value of \dot{N} in stationary state, interpretation as total scattered intensity.

Exercise: spectrum of spontaneous emission, from formal solution. Need formal solution of atomic dipole operator, Eq.(3.53).

3.3.4 Application 2: the Glauber photodetector

Roy Glauber (Nobel prize 2005) developed in the 1960s the theory of photodetection. His main result is that the signal of a photodetector is proportional to

$$I(t) \propto \langle E^{(-)}(t)E^{(+)}(t) \rangle \quad (3.49)$$

where $E^{(+)}(t)$ is the positive frequency part of the electric field operator at the detector position. This signal is constructed in such a way that if the field is in the vacuum state, a detector gives no signal: perfectly reasonable. But due to the presence of vacuum fluctuations (also nonzero expectation values of a product of field operators!), not easy to implement in the theory.

We can recover the relevant features of Glauber's theory with our two-level atom. We shall actually show that under suitable approximations,

$$I(t) \propto \langle \mathcal{E}_0^\dagger(t)\mathcal{E}(t) \rangle \quad (3.50)$$

where $\mathcal{E}(t)$ is the re-scaled electric field operator we introduced in Eq.(3.27). Note that it contains positive frequency components only, and also only those modes that are near-resonant with the atomic transition frequency ω_A . Indeed, Glauber's model for a photodetector is a two-state system that is prepared in the ground state. Incident light can be absorbed, leading to some population in the excited state. This population is then "rapidly removed" from the system. A physical example: the excited atom is ionized and the free electron moves away (it cannot come back to recombine into the ground state). This is actually the process that happens in a photomultiplier ("avalanche photodiode").

So let us see what our Heisenberg equations (3.41, 3.42) give when the atom is in the ground state and the field is in an arbitrary state. We are interested in the rate of change of the excited state population:

$$I := \frac{dp_e}{dt} = \frac{d}{dt} \left\langle \frac{\sigma_3 + \mathbb{1}}{2} \right\rangle = \frac{1}{2} \frac{d\langle \sigma_3 \rangle}{dt} \quad (3.51)$$

Since the atom is in the ground state, the operator $\sigma_3 + \mathbb{1}$ that appears in Eq.(3.41) averages to zero. We are left with

$$I = i\langle \mathcal{E}_0^\dagger(t)\sigma(t) - \sigma^\dagger(t)\mathcal{E}_0(t) \rangle \quad (3.52)$$

and insert the formal solution for the atomic dipole operator [similar to Eq.(3.24)]:

$$\sigma(t) = \sigma(0) e^{-(\gamma+i\omega_A)t} + i \int_0^t dt' \sigma_3(t')\mathcal{E}_0(t') e^{-(\gamma+i\omega_A)(t-t')} \quad (3.53)$$

This gives

$$\begin{aligned}
I &= - \int_0^t dt' \left[\langle \mathcal{E}_0^\dagger(t) \sigma_3(t') \mathcal{E}_0(t') \rangle e^{-(\gamma+i\omega_A)(t-t')} + \text{h.c.} \right] \\
&= \int_0^t d\tau \left[\langle \mathcal{E}_0^\dagger(t) \mathcal{E}_0(t-\tau) \rangle e^{-(\gamma+i\omega_A)\tau} + \text{h.c.} \right]
\end{aligned} \tag{3.54}$$

where in the second line, we used the approximation that the inversion $\sigma_3(t')$ is evolving slowly and took its expectation value in the initial state (atom in the ground state, not correlated with the field). If we also assume that the expectation value evolves slowly with time t , $\langle \mathcal{E}_0^\dagger(t) \mathcal{E}_0(t-\tau) \rangle \approx \langle \mathcal{E}_0^\dagger(t+\tau) \mathcal{E}_0(t) \rangle$ [this is strictly true for a stationary field, but stationary fields do not give not very interesting signals on a photodetector], we can combine the ‘+h.c.’ term into the single integral

$$I = \int_{-t}^t dt' \langle \mathcal{E}_0^\dagger(t) \mathcal{E}_0(t+\tau) \rangle e^{-\gamma|\tau|+i\omega_A\tau} \tag{3.55}$$

We observe that the photodetector signal is similar to the Fourier transform of the two-time field correlation function. We can already confirm that the signal is given by a normally ordered expectation value of field operator. Let us consider two limits.

Narrow-band detector

If the decay time $1/\gamma$ is “very long” (compared to the correlation time of the field), the photodetector signal is essentially given by the field spectrum, taken at the transition frequency ω_A . Indeed, the τ -integral in Eq.(3.55) picks from the positive frequency operator $\mathcal{E}_0(t+\tau)$ those components that evolve like $e^{-i\omega_k\tau}$. A narrow band photodetector is thus simply a ‘spectrometer’, and the quantity it measures is the Fourier transform of the temporal correlation (or coherence) function. Experimentally, this situation can be achieved by placing a narrow-band frequency filter in front of a photodetector.

Fast detector

In some photodetectors, the excited state has a very short lifetime $1/\gamma$. This happens for example when the excited state actually ionizes and gives away its

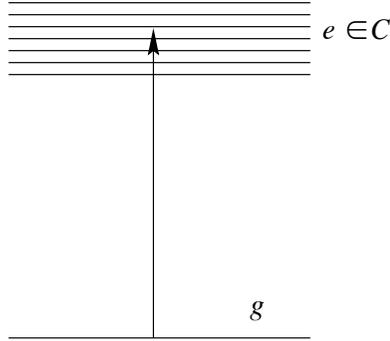


Figure 3.3: Two-level model for a fast photodetector. The signal corresponds to a sum of transition rates into all components of the upper state (quasi)continuum.

electron. The excitation rate I is then actually an integral over all components of the excited state ‘continuum’, as sketched in Fig.3.3. Under the integral, the exponential $e^{-\gamma|\tau|}$ becomes very short-ranged in τ , and we can make the replacement

$$e^{-\gamma|\tau|} \mapsto \frac{2}{\gamma} \delta(\tau) \quad (3.56)$$

This leads to Glauber’s formula

$$I(t) = \eta \langle \mathcal{E}_0^\dagger(t) \mathcal{E}_0(t) \rangle \quad (3.57)$$

where the expectation value of the instantaneous intensity operator $\mathcal{E}_0^\dagger(t) \mathcal{E}_0(t)$ appears, in normal order, of course. The prefactor η actually is a number characteristic for the detector and is called “quantum efficiency”. It must be determined experimentally. If the intensity is scaled to “photons per second”, then η gives the detection probability per photon, and I the “rate of detected photons per second”.

Intensity (photon) correlations

Generalization to multiple detection events: $I(t)$ interpreted as probability per unit time to detect one photon at time t . Probability to detect one photon at time t_1 and another one at t_2 :

$$I(t_1, t_2) = \eta^2 \langle \mathcal{E}^\dagger(t_1) \mathcal{E}^\dagger(t_2) \mathcal{E}(t_2) \mathcal{E}(t_1) \rangle, \quad t_2 > t_1 \quad (3.58)$$

We dropped the subscript 0 from $\mathcal{E}(t)$ which is still the (positive-frequency component of the) “freely evolving” electric field operator. And we also considered

that the two-photon detector is “broad band” so that just two detection times t_1 and t_2 appear. Note the time- and normal-ordered operator product. Interpretation: destroy a first photon by the action of $\mathcal{E}(t_1)$, then absorb a second photon with $\mathcal{E}(t_2)$. Square the resulting state to get the detection rate.