Chapter 6

Preparation and Measurement

The sender – Alice, say – attempts to encode a message into the state of a qubit. Quantum mechanically she must *prepare* a qubit in a certain state. The receiver – Bob, say – attempts to extract the message from the qubit received. Quantum mechanically, he must *measure* the state of the qubit.[1]

For a qubit, preparation and measurement are most transparently discussed in terms of the Pauli operators and the Stern-Gerlach magnet.

[1]Conceptually, there is no clear distinction between preparation and measurement. Indeed, according to the reproducibility assumption of the natural sciences, the preparation of a target state $\psi\rangle$, say, means nothing but that any test-for-ψ, which immediately follows the preparation, must unambiguously produce the answer "Yes". And vice-versa: reading off "Yes" in a test-for-ψ measurement means that the corresponding particle is in state $|\psi\rangle$.

6.1 Pauli operators

For the ease of presentation, we shall assume the qubit to be realized by the spin of a spin-$\frac{1}{2}$ particle. For such a system the Pauli spin operator $\hat{\vec{\sigma}} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$ is of paramount importance. Its cartesian components obey angular momentum commutation relations

$$[\hat{\sigma}_x, \hat{\sigma}_y] = 2i\hat{\sigma}_z\,, \qquad \text{with } xyz \text{ cyclic}\,, \tag{6.1}$$

and the specific spin-$\frac{1}{2}$ anti-commutation relations

$$\{\hat{\sigma}_i, \hat{\sigma}_j\} = 2\delta_{ij}\hat{1}\,, \qquad i,j = x,y,z\,. \tag{6.2}$$

Here $\hat{1}$ is the identity operator on $\mathcal{H}_{\text{qubit}}$, which we shall occasionally denote $\hat{\sigma}_0 \equiv \hat{1}$.

The set of Pauli operators, complemented by the identity, provides a basis of the space of operators $\mathcal{B}(\mathcal{H}_{\text{qubit}})$, i.e. any linear operator $\hat{A} \in \mathcal{B}(\mathcal{H}_{\text{qubit}})$ can be expanded

$$\hat{A} = \sum_{i=0,x,y,z} a_i\hat{\sigma}_i\,, \qquad a_i \in \mathbb{C}\,. \tag{6.3}$$

The operator $\hat{A}$ is self-adjoint iff $a_i \in \mathbb{R}$. For given $\hat{A}$ the expansion coefficients are easily computed,

$$a_i = \frac{1}{2}\text{tr}\{\hat{\sigma}_i\hat{A}\}\,, \qquad i = 0,x,y,z\,. \tag{6.4}$$

Acting in a two-dimensional Hilbert space, the Pauli operators admit a representation in terms of 2×2 matrices. The standard representation reads

$$\hat{\sigma}_0 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \hat{\sigma}_x \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \hat{\sigma}_y \mapsto \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \hat{\sigma}_z \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{6.5}$$

Matrix representations are quite useful – their two-dimensional lay out is better adapted to human parallel processing than the serial lay out of the abstract notation.

Adopting the convention

$$|\mathtt{0}\rangle \equiv |\uparrow_z\rangle \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad |\mathtt{1}\rangle \equiv |\downarrow_z\rangle \mapsto \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{6.6}$$

the basis vectors $|\mathtt{0}\rangle$, $|\mathtt{1}\rangle$ are identified the eigenvectors of $\hat{\sigma}_z$, the superposition vectors $|\pm\rangle$ the eigenvectors of $\hat{\sigma}_x$,

$$|+\rangle \equiv |\uparrow_x\rangle \mapsto \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad |-\rangle \equiv |\downarrow_x\rangle \mapsto \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \tag{6.7}$$

and the eigenvectors of $\hat{\sigma}_y$

$$|\uparrow_y\rangle \mapsto \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}, \qquad |\downarrow_y\rangle \mapsto \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix}. \tag{6.8}$$

6.2 Stern-Gerlach magnet

From elementary quantum mechanics we recall that the a-component of the Pauli spin, $\hat{\sigma}_a \equiv \vec{a} \cdot \hat{\vec{\sigma}}$, is measured by a Stern-Gerlach magnet (SGM) with orientation $\vec{a}$, where $\vec{a} \in \mathbb{R}^3$ is a spatial unit vector, $|\vec{a}| = 1$. The SGM is characterized by two out channels which are labeled by the eigenvalues $\sigma = \pm 1$ of $\hat{\sigma}_a$,

$$\hat{\sigma}_a |\uparrow_a\rangle = +|\uparrow_a\rangle, \qquad \hat{\sigma}_a |\downarrow_a\rangle = -|\downarrow_a\rangle. \tag{6.9}$$

Indeed, due to the master identity $\hat{\sigma}_a^2 = \hat{1}$ the eigenvalues of $\hat{\sigma}_a$ are of unit modulus, and since $\hat{\sigma}_a$ is self-adjoint, they are given by $\sigma = \pm 1$. Since $\hat{\sigma}_a$ does not commute

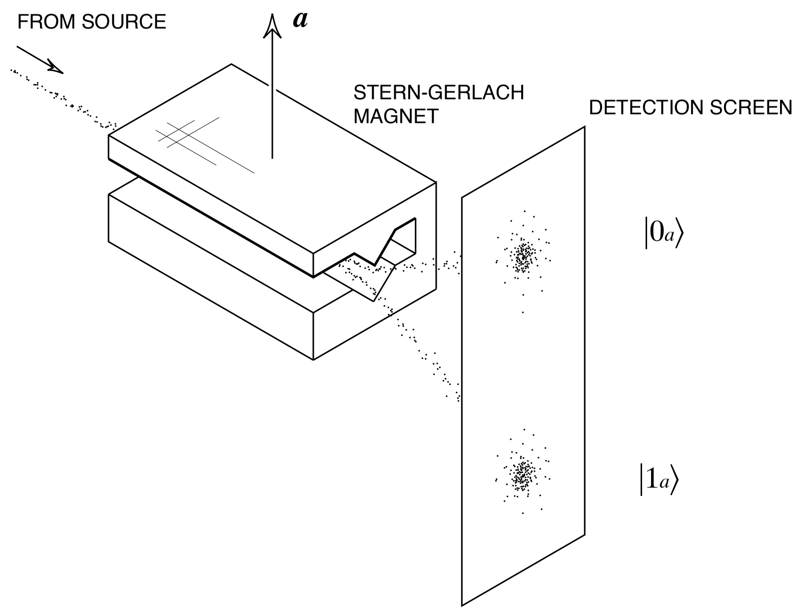

Figure 6.1: The Stern-Gerlach magnet with orientation $\vec{a}$.

with all of the Pauli operators, the eigenvalues can not be both +1 or both −1, i.e. the spectrum of $\hat{\sigma}_a$ is given by $\{+1, -1\}$.

Qubits which are detected in the upper channel (eigenvalue $\sigma = +1$) are necessarily in state $|\uparrow_a\rangle$, qubits which are detected in the lower channel (eigenvalue $\sigma = -1$) are necessarily in state $|\downarrow_a\rangle$. Tertium non datur, i.e. for any given unit vector $\vec{a}$ the pair of states $\{|\uparrow_a\rangle, |\downarrow_a\rangle\}$ form an orthonormal basis of the qubit Hilbert space

$$\langle\uparrow_a | \uparrow_a\rangle = \langle\downarrow_a | \downarrow_a\rangle = 1 \,, \quad \langle\uparrow_a | \downarrow_a\rangle = 0 \,, \tag{6.10}$$

$$|\uparrow_a\rangle\langle\uparrow_a| + |\downarrow_a\rangle\langle\downarrow_a| = \hat{1}\,. \tag{6.11}$$

As the states $|\uparrow_a\rangle$ and $|\downarrow_a\rangle$ are orthogonal, and therefore fully distinguishable, the SGM realizes a *maximal test* for the qubit: interrogation of any measurement result can not reveal more than what is already known.

Utilizing the spectral representation of $\hat{\sigma}_a$

$$\hat{\sigma}_a = |\uparrow_a\rangle\langle\uparrow_a| - |\downarrow_a\rangle\langle\downarrow_a|\,, \tag{6.12}$$

we have, according to the rules of quantum mechanics,

$$\langle\hat{\sigma}_a\rangle \equiv \langle\psi|\hat{\sigma}_a|\psi\rangle = |\langle\uparrow_a|\psi\rangle|^2 - |\langle\downarrow_a|\psi\rangle^2\,. \tag{6.13}$$

Here, the quantities

$$p(\uparrow_a|\psi) := |\langle\uparrow_a|\psi\rangle|^2\,, \qquad p(\downarrow_a|\psi) := |\langle\downarrow_a|\psi\rangle|^2\,, \tag{6.14}$$

are just the probabilities for a qubit, which enters the SGM in state $|\psi\rangle$, to end up in the upper and lower channel, respectively. One says the measurement causes a *collaps*,

$$|\psi\rangle \to |\psi'\rangle = \begin{cases} |\uparrow_a\rangle & \text{if } \sigma = +1, \text{ which occurs with prob. } p(\uparrow_a|\psi)\,, \\ |\downarrow_a\rangle & \text{if } \sigma = -1, \text{ which occurs with prob. } p(\downarrow_a|\psi)\,. \end{cases} \tag{6.15}$$

If the qubits pass the SGM without being labeled by their measurement result, the process (6.15) is called a *non-selective measurement*. If only those qubits are kept which pass through the upper channel, say, the corresponding process is called a *selective measurement*.

The SGM not only serves as a maximal test, but can equally well be employed to prepare qubits in a desired state, $|\uparrow_a\rangle$, say. Just send qubits through the SGM with orientation $\vec{a}$ with the lower channel blocked. The device now turns into a filter. All qubits, which pass the filter are in state $|\uparrow_a\rangle$. Symbolically

$$|\psi\rangle \to |\psi'\rangle = |\uparrow_a\rangle\,, \tag{6.16}$$

which is nothing but a selective measurement.

Surely, a qubit which enters the SGM with orientation $\vec{a}$ in state $|\psi\rangle = |\uparrow_a\rangle$ will leave the SGM through the upper channel *with certainty.* If send, however, through a SGM with orientation $\vec{b}$, we have

$$\langle\uparrow_a|\hat{\sigma}_b|\uparrow_a\rangle = \vec{a}\cdot\vec{b}\,, \tag{6.17}$$

from which we obtain the *transition probabilities*

$$p(\uparrow_b|\uparrow_a) \equiv |\langle\uparrow_b|\uparrow_a\rangle|^2 = \tfrac{1}{2}\left(1+\vec{a}\cdot\vec{b}\right)\,, \tag{6.18}$$

$$p(\downarrow_b|\uparrow_a) \equiv |\langle\downarrow_b|\uparrow_a\rangle|^2 = \tfrac{1}{2}\left(1-\vec{a}\cdot\vec{b}\right)\,. \tag{6.19}$$

Indeed, by definition $\langle\uparrow_a|\hat{\sigma}_a|\uparrow_a\rangle = 1$ and $\hat{\sigma}_a = \vec{a}\cdot\hat{\vec{\sigma}}$ with $\vec{a}\cdot\vec{a} = 1$, hence $\langle\uparrow_a|\hat{\vec{\sigma}}|\uparrow_a\rangle = \vec{a}$; multiplying both sides of this equation with the unit vector $\vec{b}$ proves Eq. (6.17).

Evidently, then, the SGM can also be used to build a random number generator with arbitrary bias. Noteworthy, the generator needs no special computing device, functions, or the like, but only nature and the inherent randomness of quantum mechanics.

