Chapter 8

Qubit manipulation and control

Quite frequently, one must be able to manipulate the state of a quantum system, a qubit say, in a controlled fashion. Quantum mechanically, the best we can achieve is a unitary mapping,

$$|\mathrm{out}\rangle = \hat{U}|\mathrm{in}\rangle \tag{8.1}$$

which transforms a given in-state $|\mathrm{in}\rangle$ into a desired out-state $|\mathrm{out}\rangle$. And quantum mechanically, the only way to realize this mapping is in terms of the temporal evolution of the system.

8.1 Qubit Schrödinger equation

From elementary quantum mechanics we recall that, between preparation and measurement, the temporal evolution of a quantum system, a qubit say, is governed by the Schrödinger

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle\,, \tag{8.2}$$

where $\hat{H}(t)$ is the (possibly time dependent) qubit Hamiltonian.

For a spin-$\frac{1}{2}$ particle with gyromagnetic ratio γ, the magnetic interaction energy reads $\hat{H}(t) = -\hat{\vec{\mu}}\cdot\vec{B}(t)$ where $\hat{\vec{\mu}} = \frac{\hbar\gamma}{2}\hat{\vec{\sigma}}$ is the magnetic moment of the qubit. Hence

$$\hat{H}(t) = \hbar\frac{\Omega(t)}{2}\vec{b}(t)\cdot\hat{\vec{\sigma}}\,. \tag{8.3}$$

where $\vec{b}(t) = \frac{\vec{B}(t)}{|\vec{B}(t)|}$ the (momentary) direction of $\vec{B}(t)$, and $\Omega(t) = -\gamma|\vec{B}(t)|$ the (momentary) Larmor frequency.

Consider a qubit which traverses, with fixed velocity V, a homogeneous magnetic field of linear extension L. Denote $|\mathrm{in}\rangle$ the qubit state at the entrance to the interaction region, the qubit state upon leaving the interaction region after an interaction time $T = L/V$ is given Eq. (8.2),

$$\hat{U} = e^{i\delta}\hat{R}_b(\theta)\,, \tag{8.4}$$

where $\delta\in\mathbb{R}$ is a phase (here $\delta = 0$), and $\hat{R}_b(\theta)$ is a *special unitary*

$$\hat{R}_b(\theta) = \exp\{-i\tfrac{\theta}{2}\vec{b}\cdot\hat{\vec{\sigma}}\}\,, \qquad \theta = \Omega T\,. \tag{8.5}$$

Observing $(\vec{b}\cdot\hat{\vec{\sigma}})^2 = \hat{1}$, we have

$$\hat{R}_b(\theta) = \cos\left(\tfrac{\theta}{2}\right)\hat{1} - i\sin\left(\tfrac{\theta}{2}\right)\vec{b}\cdot\hat{\vec{\sigma}}\,. \tag{8.6}$$

The operator $\hat{R}_b(\theta)$ generates rotations of the Bloch sphere around direction $\vec{b}$ by an angle θ. Given any pair of states $|\mathrm{in}\rangle$, $|\mathrm{out}\rangle$, (two points on the Bloch sphere), there exists a physical process (Hamiltonian) such that the associated $\hat{U}$ maps $|\mathrm{in}\rangle\mapsto|\mathrm{out}\rangle = \hat{U}|\mathrm{in}\rangle$.

8.2 Single qubit gates

The map (8.1) may be viewed the action of a single qubit **logic gate**,

$$|\mathrm{in}\rangle \sim\!\boxed{U}\!\sim |\mathrm{out}\rangle \tag{8.7}$$

Since $\hat{U}$ is unitary, the gate is **reversible**: you may infer the input from the output. Classically, there are only two reversible single bit gates: the identity ID, and the NOT. Quantum mechanically, there are infinitely many gates – the whole of $U(2)$, the group of unitary 2×2 matrices.

The following unitaries provide useful gate operations

$$\text{NOT} \quad \sim\!\boxed{X}\!\sim \quad , \quad \hat{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \tag{8.8}$$

$$\text{HADAMARD} \quad \sim\!\boxed{H}\!\sim \quad , \quad \hat{H} = \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} . \tag{8.9}$$

The NOT flips the qubit state, the HADAMARD generates superpositions of the computational basis states. Note that for both gates $\text{HADAMARD}^2 = \text{NOT}^2 = \text{ID}$.

Most quantum gates have no classical counterpart. Consider the square-root of NOT,

$$\sqrt{\text{NOT}} = \tfrac{1}{\sqrt{2i}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} . \tag{8.10}$$

Obviously $\sqrt{\text{NOT}}\sqrt{\text{NOT}} = \text{NOT}$, but there is no *classical* gate $f : \{0,1\} \mapsto \{0,1\}$ such that $f \circ f = \text{NOT}$.

Concatenation of two quantum gates can be viewed an *interferometer* – see Fig. 8.1 – and vice versa: any interferometer may be viewed the concatenation of two or more quantum logical gates.

Denoting $\psi_{\mu\nu} := \langle\mu|\sqrt{\text{NOT}}|\nu\rangle$, with $\mu,\nu \in \{0,1\}$ the transition amplitudes of $\sqrt{\text{NOT}}$, the probability to end up in state $|f\rangle$ is given by

$$
\begin{aligned}
|\langle f|\text{NOT}|i\rangle|^2 &= |\langle f|\sqrt{\text{NOT}}\sqrt{\text{NOT}}|i\rangle|^2 &\quad (8.11)\\
&= \left|\psi_{f1}\psi_{1i} + \psi_{f0}\psi_{0i}\right|^2 , &\quad (8.12)
\end{aligned}
$$

where we have inserted the resolution of unity, $\hat{1} = \sum_{\mu=0,1} |\mu\rangle\langle\mu|$ between the two $\sqrt{\text{NOT}}$. On the solid loop $i = 0 \rightsquigarrow f = 1$, the two partial amplitude $\psi_{11}\psi_{10}$ and $\psi_{10}\psi_{00}$ interfere constructively, on the dashed loop $i = 0 \rightsquigarrow f = 1$ the two partial amplitudes interfere destructively.

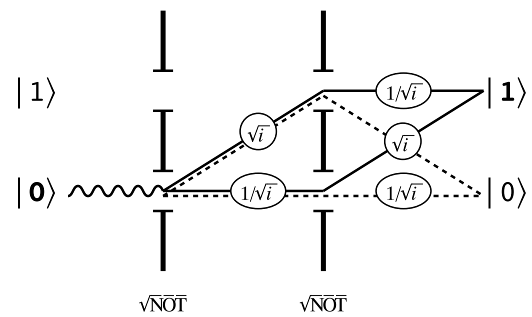

Figure 8.1: Hilbert space interferometry. The NOT viewed as concatenation of two $\sqrt{\text{NOT}}$ gates. Each of the end points on the right can be reached via two different paths. For incoming particles in state $|0\rangle$, there is constructive interference in the upper loop (solid), but destructive interference in the lower loop (dashed). Accordingly, all the particles will be measured in state $|1\rangle$, none in state $|0\rangle$. The circled numbers are the gates's *transition amplitudes* (up to proportionality $1/\sqrt{2}$), for example $\langle 1|\sqrt{\text{NOT}}|0\rangle = \sqrt{i/2} \propto \sqrt{i}$.

