Chapter 9

Composite Systems

9.1 Hilbert space of a composite system

We consider two quantum objects, each being equipped with its own Hilbert space and orthonormal basis (ONB),

$$\mathcal{H}_A\,,\quad \text{ONB } \{|u_\mu\rangle\}_{\mu=1}^{M}\,,\quad \text{general vector } |\phi\rangle = \sum_{\mu=1}^{M} \phi_\mu |u_\mu\rangle\,, \tag{9.1}$$

$$\mathcal{H}_B\,,\quad \text{ONB } \{|v_\nu\rangle\}_{\nu=1}^{N}\,,\quad \text{general vector } |\chi\rangle = \sum_{\nu=1}^{N} \chi_\nu |v_\nu\rangle\,. \tag{9.2}$$

The two objects form a composite quantum system with Hilbert space

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B\,,\qquad \text{ONB } \{|u_\mu v_\nu\rangle \equiv |u_\mu\rangle \otimes |v_\nu\rangle\}_{\mu,\nu=1}^{M,N}\,. \tag{9.3}$$

The Hilbert space of the composite system is the space of all product vectors $|\phi\rangle \otimes |\chi\rangle$ *and linear combinations thereof* (most of which can not be written as product vectors). The dimension of that space is $\dim\mathcal{H} \equiv L = M \times N$.

Definition A state vector is called a *disentangled* iff it can be written in the form of a product vector

$$\begin{aligned} |\psi\rangle &= |\phi\rangle \otimes |\chi\rangle && (9.4)\\ &= \sum_{\mu\nu} \phi_\mu \chi_\nu |u_\mu\rangle \otimes |v_\nu\rangle \,. && (9.5) \end{aligned}$$

A state vector is called *entangled* iff it can *not* be written in the form of a product state.

Note that product vectors do not form a linear manifold (subspace) in $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Note also that entanglement is the rule rather than the exception, as the set of product vectors is a manifold of dimension $2(M+N)-4$ while the set of all state vectors is a manifold of much larger dimension $2M \times N - 2$.

For two qubits, $M = N = 2$, the Hilbert space of the composite system is 4-dimensional, $L = 4$. Utilizing the matrix representation in the factor spaces,

$$|\phi\rangle \mapsto \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \in \mathbb{C}^2_A \,, \qquad |\chi\rangle \mapsto \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} \in \mathbb{C}^2_B \,, \tag{9.6}$$

we induce the matrix representation in the product space

$$|\phi\rangle \otimes |\chi\rangle \mapsto \begin{bmatrix} \phi_1 \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} \\ \phi_2 \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \phi_1\chi_1 \\ \phi_1\chi_2 \\ \phi_2\chi_1 \\ \phi_2\chi_2 \end{bmatrix} \in \mathbb{C}^4_{AB} \,. \tag{9.7}$$

9.2 Schmidt decomposition

By definition, every state vector of a bi-partite system can be expanded on the product basis (9.3),

$$|\psi\rangle = \sum_{\mu\nu} c_{\mu\nu} |u_\mu v_\nu\rangle\,, \tag{9.8}$$

which involves $M \times N$ complex coefficients $c_{\mu\nu}$. By means of a smart choice of basis, however, most of these coefficients turn out to be zero.

Theorem (Schmidt Decomposition) A state vector of a bi-partite system can always be expanded

$$|\psi\rangle = \sum_{\nu}^{r} \sqrt{p_\nu} |e_\nu\rangle \otimes |f_\nu\rangle\,, \tag{9.9}$$

with *Schmidt-coefficients* $p_\mu > 0$, $\sum_\mu^r p_\mu = 1$, and $|e_\nu\rangle$ and $|f_\nu\rangle$ orthonormal vectors in $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. The natural number $r \leq \min(M, N)$ is called the *Schmidt number* of $|\psi\rangle$.

The proof is straightforward. By the *singular value decomposition*, $c = udv$, where d is a diagonal matrix with non-negative elements, and u and v are unitary matrices. Thus $|\psi\rangle = \sum_{\mu\nu'\nu} u_{\mu\nu'} d_{\nu'\nu'} v_{\nu'\nu} |u_\mu\rangle|v_\nu\rangle$. Define $|e_{\nu'}\rangle = \sum_\mu u_{\mu\nu'}|u_\mu\rangle$, $|f_\nu\rangle = \sum_\nu v_{\nu'\nu}|v_\nu\rangle$, and set $d_{\nu'\nu'} = \sqrt{p_{\nu'}}$, this gives Eq. (9.9). Orthonormality of the set of vectors $|e_\nu\rangle$ from the unitarity of u, orthornormality of the set $|f_\nu\rangle$ follows from the unitarity of v. In practice, u and v are constructed by diagonalizing the hermitian(!) $cc^\dagger$ and $c^\dagger c$, respectively. *End-of-proof*

The Schmidt decomposition is quite a powerful tool in the analysis of bi-partite systems. For example

Theorem A state vector is entangled iff its Schmidt number exceeds unity.

No proof necessary *End-of-proof*

9.3 Bases for two qubits

The basis system displayed in (9.3) is a special type of basis, called *product basis*. For the case of two qubits, the most popular product basis consists of common eigenvectors of $\hat{\sigma}_z$ and $\hat{\tau}_z$ ($\hat{\vec{\tau}}$ is Bob's Pauli spin operator).

basis ket	eigenvalues	
	$\hat{\sigma}_z$	$\hat{\tau}_z$
$\vert\uparrow\uparrow\rangle$	$+1$	$+1$
$\vert\uparrow\downarrow\rangle$	$+1$	-1
$\vert\downarrow\uparrow\rangle$	-1	$+1$
$\vert\downarrow\downarrow\rangle$	-1	-1

Figure 9.1: A popular product basis of a two qubit system

For many purposes, however, another type of basis, called *entangled* basis turns out to be more useful. For the case of two qubits, the most popular entangled basis is the *Bell* basis. It consists of common eigenstates of $\hat{\sigma}_x\hat{\tau}_x$ and $\hat{\sigma}_y\hat{\tau}_y$ (and also $\hat{\sigma}_z\hat{\tau}_z$ because $\hat{\sigma}_z\hat{\tau}_z = -\hat{\sigma}_x\hat{\tau}_x\hat{\sigma}_y\hat{\tau}_y$).

basis ket	eigenvalues $\hat{\sigma}_x\hat{\tau}_x$	$\hat{\sigma}_y\hat{\tau}_y$	$\hat{\sigma}_z\hat{\tau}_z$
$\|\psi^+\rangle = (\|\uparrow\downarrow\rangle + \|\downarrow\uparrow\rangle)/\sqrt{2}$	$+1$	$+1$	-1
$\|\phi^+\rangle = (\|\uparrow\uparrow\rangle + \|\downarrow\downarrow\rangle)/\sqrt{2}$	$+1$	-1	$+1$
$\|\phi^-\rangle = (\|\uparrow\uparrow\rangle - \|\downarrow\downarrow\rangle)/\sqrt{2}$	-1	$+1$	$+1$
$\|\psi^-\rangle = (\|\uparrow\downarrow\rangle - \|\downarrow\uparrow\rangle)/\sqrt{2}$	-1	-1	-1

Figure 9.2: The Bell basis of a two qubit system

9.4 Operators for two qubits

Denote the Pauli operators for Alice's and Bob's qubit $\hat{\sigma}_i$ and $\hat{\tau}_i$, respectively. A linear operator $\hat{X}$ on $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ can be expanded in the form

$$\hat{X} = \sum_{i,j=0,x,y,z} x_{ij}\hat{\sigma}_i \otimes \hat{\tau}_j \,. \tag{9.10}$$

The operator $\hat{X}$ is self-adjoint iff all the coefficients x_{ij} are real.

The matrix representation (9.6) induces a matrix representation of operators. For $\hat{A} \in \mathcal{B}(\mathcal{H}_A)$ and $\hat{B} \in \mathcal{B}(\mathcal{H}_B)$ with matrix representations

$$\hat{A} \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad \hat{B} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \tag{9.11}$$

we have, for $\hat{A} \otimes \hat{B} \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$,

$$\hat{A} \otimes \hat{B} \mapsto \begin{bmatrix} a\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} & b\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \\ c\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} & d\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \end{bmatrix} = \begin{bmatrix} a\alpha & a\beta & b\alpha & b\beta \\ a\gamma & a\delta & b\gamma & b\delta \\ c\alpha & c\beta & d\alpha & d\beta \\ c\gamma & c\delta & d\gamma & d\delta \end{bmatrix}. \tag{9.12}$$

9.5 Entanglement and Correlations

Entangled state vectors give rise to *correlations* (or anti-correlations) of Alice's and Bob's measurement results. If Alice and Bob share a pair of qubits in the entangled state $|\psi^-\rangle$, say, and both perform a Stern-Gerlach measurement with the same orientation, $\vec{a} = \vec{b}$, they will find that whenever Alice's particle leaves her device through the upper channel (lower channel) Bob's particle will leave his device through the opposite channel.

Product states, in contrast, do not give rise to correlations. Consider the situation where Alice measures $\hat{A}$ and Bob measures $\hat{B}$ on a bi-partite system. This measurement is described by the observable $\hat{A} \otimes \hat{B}$, and for a pure product state $|\phi\rangle \otimes |\chi\rangle$, the expectation value of that observable factorizes,

$$\langle \hat{A} \otimes \hat{B} \rangle = \langle \phi|\hat{A}|\phi\rangle \, \langle \chi|\hat{B}|\chi\rangle \, . \tag{9.13}$$

Surely, sharing a pair of qubits in a product state, Alice can not communicate with Bob by just manipulating her qubit. But what if Alice and Bob share a pair of entangled qubits? Again the answer is "No!" – the Bell telephone is also an impossible machine.

