Chapter 10

EPR Paradox and Bell inequalities

In a landmark paper entitled "Can quantum mechanical description of physical reality be considered complete" [Phys. Rev. **47**, 777 (1935)] the authors Einstein, Podolsky and Rosen (EPR) employ entanglement to argue against the Copenhagen interpretation of quantum mechanics. The authors base their argument on the entanglement of the motional degrees of freedom of two point particles. Here we follow David Bohm who has reformulated the argument in terms of two spin-$\frac{1}{2}$ particles.[1]

Assume Alice and Bob, who are distantly apart, share a pair of qubits in the Bell state $|\psi^-\rangle$. As this state vector transforms as a singlet under rotations we have for any two directions $\vec{c}$, $\vec{c}'$

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_c\downarrow_c\rangle - |\downarrow_c\uparrow_c\rangle\right) = \frac{1}{\sqrt{2}}\left(|\uparrow_{c'}\downarrow_{c'}\rangle - |\downarrow_{c'}\uparrow_{c'}\rangle\right) . \qquad (10.1)$$

Alice and Bob perform measurements on their individual qubit, each using a SGM

[1] *Quantum Theory* by D. Bohm, Prentice Hall, Eaglewood Cliffs, New Jersey 1951.

with its own orientation $\vec{a}$ and $\vec{b}$ respectively. We denote $A = \pm 1$ the spin-value (outcome) of Alice's qubit, and $B = \pm 1$ the spin-value (outcome) of Bob's qubit.

10.1 The facts

The following are the only facts which we need in the subsequent exposition of the EPR argument:

(i) For any action undertaken by Bob, and for any orientation $\vec{a}$ of Alice's SGM, her qubit leaves the SGM either through the upper channel ($A = +1$) exclusive-or through the lower channel ($A = -1$). Similarly for Bob.

(ii) If both parties agree on the same orientation $\vec{a} = \vec{b}$, the measurement results are strictly anti-correlated $AB = -1$.

(iii) For any orientation of her SGM, and any action undertaken by Bob, Alice's qubits turn $A = \pm 1$ with equal probability. Similarly for Bob.

There are many more facts associated with the Bell state, but these are irrelevant for the EPR argument, the Bell theorem, and the clash of quantum mechanics with local realism.

10.2 The Paradox

Suppose for a given pair, Alice chooses to measure $\hat{\sigma}_z$ on her particle, and she finds $A = +1$. She then knows that if Bob would measure $\hat{\tau}_z$, he necessarily would find $B = -1$ for his particle. Yet, since Alice and Bob are far apart, her act of measurement can not have influenced the state of Bob's particle. From

> "If, without in any way disturbing a system, we can predict with certainty the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity. This means that this physical quantity has a value independently of whether we measure it or not",

EPR infer that Bob's spin component $\hat{\tau}_z$ is an *element of physical reality* (has a definite value).

Suppose Alice chooses to measure $\hat{\sigma}_x$, and she finds $A = -1$. Now she knows that if Bob would measure $\hat{\tau}_x$, he necessarily would find $B = +1$. According to EPR, Bob's spin component $\hat{\tau}_x$ is also an element of physical reality (has a definite value).

EPR thus arrive at the paradox that both $\hat{\tau}_x$ and $\hat{\tau}_z$ are elements of physical reality, i.e. they both have definite values, but according to quantum mechanics they can not have both definite values as the corresponding operators do not commute. Hence

> "It becomes evident that the paradox forces us to relinquish one of the following assertions: (1) the description by means of a wave function is not complete; (2) the real states of spatially separated systems are not independent."

According to EPR, the second assertion is indisputable since

> "[...] the situation of system B is independent of what is done with system A, which is spatially separated from the former".

This conviction has subsequently been dubbed *principle of Einstein locality*.

EPR assert that (1) holds, i.e. they conclude that quantum mechanics is not complete. According to EPR there are additional variables, which so far could neither be qualified nor measured, that determine the result of any particular experiment. However averaging over the possible values of these *hidden variable*, one would obtain the same result as those predicted by ordinary quantum mechanics. A *realistic theory* is that which assumes that the observables can be described by a set of hidden variables.

10.3 Bohr response

According to Bohr, the EPR argument does not hold as no conclusion about reality may be drawn from incompatible measurements. A measurement of $\hat{\sigma}_z$ tells all about a measurement of $\hat{\tau}_z$, yet it has nothing to say about a measurement of $\hat{\tau}_x$. To say something about the latter, one needs a different experimental set-up, and this has to be analyzed separately.

10.4 Bell theorem

John Stuart Bell, in the most influential paper *On the Einstein-Podolsky Rosen paradox*, Physics **1**, p. 195–200 (1964),[2] proves the following

Theorem Einstein-local hidden variable theories are incompatible with quantum

[2] Reprinted in: *Speakable and Unspeakable in Quantum Mechanics* by J. S. Bell, Cambridge University Press, Cambridge 1987

mechanics, i.e. they make different predictions.

For a proof we shall analyze Alice's and Bob's spin measurements on the EPR pairs in the singlet state $|\psi^-\rangle$.

Recall the *facts*: the measurement of $\hat{\sigma}_a$ on Alice's qubit yields either $A = +1$ or $A = -1$. Similarly, measurement of $\hat{\sigma}_b$ on Bob's qubit yields $B = \pm 1$. For equal alignment, measurement of each pair yields $AB = -1$.

In any Einstein-local hidden variable theory of the *facts*, the variable A is a dichotomic variable which depends on (i) the direction $\vec{a}$, and (ii) a set of hidden variables λ, that is $A = A(\vec{a}, \lambda) = \pm 1$. Similarly $B = B(\vec{b}, \lambda) = \pm 1$. Due to Einstein locality, neither does A depend on the Bob's direction $\vec{b}$, nor does B depend on the Alice's direction $\vec{a}$. However, for equal orientation $\vec{a} = \vec{b}$, there is strict anti-correlation, $A(\vec{a}, \lambda)B(\vec{a}, \lambda) = -1$.[3] The hidden variables are in fact stochastic variables which are distributed according to some probability density $\rho(\lambda)$. The spin-spin correlation function is then given by

$$\mathcal{C}(\vec{a}, \vec{b}) = \int d\lambda \rho(\lambda) A(\vec{a}, \lambda) B(\vec{b}, \lambda)\,. \tag{10.2}$$

From this we obtain

$$\begin{aligned} \mathcal{C}(\vec{a}, \vec{b}) - \mathcal{C}(\vec{a}, \vec{c}) &= \int d\lambda \rho(\lambda) \Big[A(\vec{a}, \lambda)B(\vec{b}, \lambda) - A(\vec{a}, \lambda)B(\vec{c}, \lambda)\Big] && (10.3) \\ &= -\int d\lambda \rho(\lambda) A(\vec{a}, \lambda) A(\vec{b}, \lambda) \Big[1 + A(\vec{b}, \lambda) B(\vec{c}, \lambda)\Big]\,, && (10.4) \end{aligned}$$

where we have used $A(\vec{b}, \lambda)^2 = 1$, and $B(\vec{b}, \lambda) = -A(\vec{b}, \lambda)$. Taking the modulus, and

[3] A possible model is $A = \mathrm{sgn}(\vec{a} \cdot \vec{\lambda})$, $B = -\mathrm{sgn}(\vec{b} \cdot \vec{\lambda})$, where $\vec{\lambda}$ is some (stochastic) unit vector, and sgn is the signum-function (function-values ± 1).

observing $|A(\vec{a},\lambda)A(\vec{b},\lambda)| = 1$, we find the *Bell inequality*

$$|\mathcal{C}(\vec{a},\vec{b}) - \mathcal{C}(\vec{a},\vec{c})| \leq 1 + \mathcal{C}(\vec{b},\vec{c})\,. \tag{10.5}$$

This inequality holds for any local hidden variable theory which attempts to model the facts.

According to quantum mechanics, on the other hand, the spin-spin correlation function is given by

$$\mathcal{C}_{\mathrm{QM}}(\vec{a},\vec{b}) = \langle\psi^-|\hat{\sigma}_a \otimes \hat{\tau}_b|\psi^-\rangle = -\vec{a}\cdot\vec{b} \tag{10.6}$$

Choosing $\angle(\vec{a},\vec{b}) = \angle(\vec{b},\vec{c}) = \pi/3$, $\angle(\vec{a},\vec{c}) = 2\pi/3$, we find $\mathcal{C}_{\mathrm{QM}}(\vec{a},\vec{b}) = -\frac{1}{2}$, $\mathcal{C}_{\mathrm{QM}}(\vec{a},\vec{c}) = +\frac{1}{2}$, and thus $|\mathcal{C}_{\mathrm{QM}}(\vec{a},\vec{b}) - \mathcal{C}_{\mathrm{QM}}(\vec{a},\vec{c})| = 1 > \frac{1}{2} = 1 + \mathcal{C}_{\mathrm{QM}}(\vec{b},\vec{c})$ in *violation* of the Bell inequalities. The quantum mechanical correlations are stronger than in any “classical” theory!

10.5 The solution of the paradox

EPR use a special type of argument, called *counterfactual* argument. “Alice measures $\hat{\sigma}_z \ldots$, but if – for the *same* particle – she *would* measures $\sigma_x \ldots$.” In quantum mechanics you may speculate what would happen if, but you must exercise great care in drawing the conclusions.

Einstein-local hidden variable theories are not only incompatible with the predictions of quantum mechanics, but they contradict the experiments: measurements, first carried out by Clauser, and subsequently by Aspect and collaborators, clearly demonstrate a violation of the Bell inequalities. Nature proves quantum mechanics right. The world is just *not* locally realistic.

That does not mean that the world is not causal a place. The EPR correlations can not be used to make superluminal phone calls. Whatever Alice does with her qubit, the outcome on Bob's side is completely random. It is only in comparing the measurement records that the correlations become manifest. As this comparism requires the exchange of data, it's speed is bound by the velocity of light. The *Bell telephone*, that is the entanglement-assisted superluminal communication, is just another *impossible machine*.