Chapter 12

Entanglement is the rule

You may have the impression that entanglement is rather special, but this is not the case. In fact, entanglement is not the exception but the rule, and we usually have problems with "too much" entanglement. Any interaction of a qubit, say, with its environment results in an entangled state: entanglement, however, which we can not exploit since we usually can not control all the degrees of freedom of the environment.[1]

Consider spontaneous emission. If initially the atom is in its excited state, and the electromagnetic field is in its vacuum state, after some time t the state of the combined system "atom+field" will have evolved

$$|\Psi(t)\rangle = c_0(t)|e\rangle \otimes |0\dots 0\rangle + \sum_i c_i(t)|g\rangle \otimes |0\dots 01_i0\dots 0\rangle\,, \tag{12.1}$$

where $|0\dots 01_i0\dots 0\rangle$ denotes the state of the electromagnetic field with one photon

[1] This lecture was skipped in the WS00 course

in mode i, all other modes in the vacuum state. The exact numerical values of the coefficients $c_i(t)$ may be difficult to come by, but for sure Eq. (**??**) is a highly entangled state. Yet, as we can not control each and every of the electromagnetic field modes i individually, this entanglement is of little use. In fact, it not only is of little use, but rather detrimental.

12.1 Entanglement and measurement

[This needs some more work]

We consider a Young double slit experiment for spin-$\frac{1}{2}$ particle. We assume some magnetic field behind one of the slits – the slit on the right side, say. As we shall see the magnetic field – although it has no impact on the motional state of the atoms – may have a severe impact on the interference pattern of the double slit.

The particles impinge in some definite spin state $|\uparrow_z\rangle$. After passage through the double slit and just right before detection on the screen (which is positioned in the far field of the double slit), the state vector reads

$$|\Psi\rangle = \frac{|l\rangle \otimes |\uparrow_z\rangle + |r\rangle \otimes |\uparrow_a\rangle}{\sqrt{2}}, \qquad (12.2)$$

where $|l\rangle$ ($|r\rangle$) is the translational state of particles which passed through the left (right) slit (with the other slit closed), and $\uparrow_a$ is the rotated spin state of particles which passed through the right slit.

The state vector (12.2) is that of an entangled state, yet it is not written in the form of a Schmidt decomposition. In contrast to the motional state "passage though the left slit" $|l\rangle$ and "passage through the right slit" $|r\rangle$, which are true alternatives, $\langle l|r\rangle = 0$, the spin state are not necessarily orthogonal, i.e. $\langle\uparrow_z \mid \uparrow_a\rangle \neq 0$. The

Schmidt decomposition of Eq. (12.2) reads

$$|\Psi\rangle = \sqrt{p}|\phi_+\rangle \otimes |\uparrow_c\rangle + \sqrt{1-p}|\phi_-\rangle \otimes |\downarrow_c\rangle \tag{12.3}$$

where $\vec{c} = \vec{e}_z + \vec{a}/|\vec{e}_z + \vec{a}|$ is a spatial unit vector, $|\phi_\pm = (|r\rangle \pm |l\rangle)/\sqrt{2}$, and

$$p = \frac{1+\beta}{2} \text{ with } \beta = \langle \uparrow_z | \uparrow_a \rangle \,. \tag{12.4}$$

The particles impinge on a detection screen (a CCD camera, say) which is not sensitive to the spin state. The probability density I at point P on the detection screen is given by

$$I(P) \propto |l(P)|^2 + |r(P)|^2 + \beta l(P)^* r(P) + \beta r(P)^* l(P) \,. \tag{12.5}$$

The density I diplays an interference pattern, the modulation depth of which, called **fringe contrast**,

$$\gamma := \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} \tag{12.6}$$

depends sensitively on the spin-state overlap β. From the definition of the fringe contrast we find

$$\gamma = |\beta| \,. \tag{12.7}$$

For maximally distinguishable spin states $\beta = 0$, the contrast is zero and the interference pattern turns into the distribution of classical point particles. For indistinguishable spin states, $\beta = 1$, contrast is maximal, i.e. the density distribution is "maximally quantum". Note that the reduction in contrast takes place even though, in our model, the particle's motion is not influenced by the magnetic field.

Yet spin measurement may reveal information about the slit each invidual particle took before reaching the screen, called **which-path-information**. The more which-path information we can extract, the lower the fringe contrast, and concomitantly, the more "classical" the density distribution.

Rules of elementary quantum mechanics dictate

$$\mathrm{Prob}(l,\uparrow_z) = \tfrac{1}{2}\,, \qquad \mathrm{Prob}(r,\uparrow_z) = \tfrac{1}{2}q\,, \tag{12.8}$$

$$\mathrm{Prob}(l,\downarrow_z) = 0\,, \qquad \mathrm{Prob}(r,\downarrow_z) = \tfrac{1}{2}\,(1-q)\;. \tag{12.9}$$

where $q = |\beta|^2$. From this we infer $\mathrm{Prob}(\uparrow_z) = \frac{1}{2}(1+q)$, $\mathrm{Prob}(\downarrow_z) = \frac{1}{2}(1-q)$, and the conditional probabilities

$$\mathrm{Prob}(l|\uparrow_z) = \frac{1}{1+q}\,, \qquad \mathrm{Prob}(r|\uparrow_z) = \frac{q}{1+q}\,, \tag{12.10}$$

$$\mathrm{Prop}(l|\downarrow_z) = 0\,, \qquad \mathrm{Prob}(r|\downarrow_z) = 1\,. \tag{12.11}$$

The conditional probabilities, in turn, can be quantified in terms of conditional entropies. The residual uncertainty, which remains about the path, given the particle is detected with spin up,

$$H(\mathrm{path}|\uparrow_z) = \frac{1}{1+q}\mathrm{ld}\,(1+q) + \frac{q}{1+q}\mathrm{ld}\left(\frac{1+q}{q}\right)\,, \tag{12.12}$$

while $H(\mathrm{path}|\downarrow_z) = 0$, because $\downarrow_z$ can only be found for particles which took the right slit.

The initial level of ignorance about the path is $H_{\mathrm{initial}} = 1bit$. The average level of ignorance which remains after a spin measurement is $H_{\mathrm{final}} = \mathrm{Prob}(\uparrow_z)H(\mathrm{path}|\uparrow_z) + \mathrm{Prob}(\downarrow_z)H(\mathrm{path}|\downarrow_z)$. The average information gain $I_{\mathrm{av}} = H_{\mathrm{initial}} - H_{\mathrm{final}}$

$$I_{\mathrm{av}} = 1 - \frac{1}{2}\mathrm{ld}\,(1+q) - \frac{q}{2}\mathrm{ld}\left(\frac{1+q}{q}\right)\,. \tag{12.13}$$

Information gain is maximal if the spin states $\uparrow_z$, $\uparrow_a$ are orthogonal, $q = 0$. If the spin states are parallel, $q = 1$, information gain is zero – in this case spin measurement does not reveal any which-path-information.

Evidently, it is the mere presence of which-path information, and not the uncontrolled scattering of a photon, say, which affects the spatial density distribution. The more we can learn about the path, the more classical appears the distribution. The less we can learn about the path, the more quantum appears the distribution.

In our model, the which-path information and fringe contrast is intimately linked to the entanglement between the motional and spin degrees of freedom. The more entanglement, the better the which-path measurement. The better the which-path measuremnt, the less quantum the pattern. The less quantum the pattern, the more classical the distribution. Entanglement may well destroy that what is most important – the coherence. In the present case it destroys the coherence between the wave functions $l(x)$ and $r(x)$.