Chapter 13

Quantum Mechanics 2.0

Any Hilbertspace element $|\psi\rangle$ of unit norm characterizes a particular physical state ψ of a quantum system, from now on called a pure state. The preparation of a pure state, however, is an ideal rarely ever realized. Quite common, preparation devices come with some uncontrollable disturbances, some shaking of the lab table, interventions of cosmic particle shower, microwave background, stray photons, and even intentional interventions, like switching for this or for that, with the switch depending, say, on the outcome of the experimenter rolling a die.

Example Consider a memoryless source which prepares qubits in pure states $|\psi_i\rangle$ with probability (relative frequency) p_i, $i = 1, \ldots, N$. The number of different state N may be arbitrary, even greater then $\dim\mathcal{H}$, and the $|\psi_i\rangle$ must only be normalized, $\langle\psi_i|\psi_i\rangle = 1$, but must not be mutually orthogonal. According to the rules of statistical physics, the expectation value of an observable $\hat{A}$, say,

is given by

$$\langle \hat{A} \rangle = \sum_{i=1}^{N} p_i \langle \psi_i | \hat{A} | \psi_i \rangle \,. \tag{13.1}$$

Introducing the operator

$$\hat{\varrho} = \sum_{i=1}^{N} p_i |\psi_i\rangle\langle\psi_i| \,, \tag{13.2}$$

Eq. (13.1) can also be written

$$\langle \hat{A} \rangle = \mathrm{tr}\{\hat{A}\hat{\varrho}\} \,. \tag{13.3}$$

A glance on Eq. (13.1) reveals that the statistics of the measured values on the qubit is governed by two distinct sources. The one source being the statistical nature of quantum mechanics (which comes with each of the pure states $|\psi_i\rangle$), the other being the statistical character of the preparation (which comes with the "classical" probabilities p_i).

The operator (13.2) is a particular instances of a so called *state operator*, i.e. an operator on $\mathcal{H}$ which is self-adjoint, positive and normalized trace one.

Replacing the notion of a state as being described by a Hilbert-space vector by a self-adjoint operator which acts on Hilbertspace requires some adaption of the original quantum mechanical postulates – that is (i) the state postulate, (ii) the observable postulate, and (iii) the time evolution postulate. We start with the state postulate.

13.1 The QM 2.0 Postulate of State

Postulate I (State Postulate) A **state** of a quantum mechanical system with Hilbert space $\mathcal{H}$ is given by a linear operator $\hat{\varrho} : \mathcal{H} \to \mathcal{H}$ which is

$$(s1) \quad \hat{\varrho} = \hat{\varrho}^\dagger \quad \text{self adjoint} \tag{13.4}$$

$$(s2) \quad \mathrm{Tr}(\hat{\varrho}) = 1 \quad \text{normalized} \tag{13.5}$$

$$(s3) \quad \langle\psi|\hat{\varrho}|\psi\rangle \geq 0 \quad \text{positiv semi-definit} \tag{13.6}$$

Being normalized and positiv semi-definit, the set of states is a subset, denoted $\mathcal{S}(\mathcal{H})$, of all bound operators on $\mathcal{H}$, which is commonly denoted $\mathcal{B}(\mathcal{H})$. In short $\mathcal{S}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$. Equipped with the rule for "operator linear combination" $(a\hat{A} + b\hat{B})|\psi\rangle = a(\hat{A}|\psi\rangle + b(\hat{B}|\psi\rangle)$, the set $\mathcal{B}(\mathcal{H})$ is promoted a linear space. On this space, the map $\|\cdot\| : \mathcal{B} \to \mathbb{R}$, with $\|\hat{A}\|^2 = \mathrm{Tr}(\hat{A}^\dagger\hat{A})$ obeys all requirements for a norm, i.e. $\mathcal{B}(\mathcal{H})$ is a **Banach space**. Note that state space $\mathcal{S}(\mathcal{H})$, albeit a subset of $\mathcal{B}(\mathcal{H})$, is not a linear space on its own. With $\hat{\varrho}_1$ and $\hat{\varrho}_2$ both in $\mathcal{S}(\mathcal{H})$, the sum $\hat{\varrho}_1 + \hat{\varrho}_2$ is clearly not element of $\mathcal{S}(\mathcal{H})$. For example, the sum is just not normalized, check the postulate (s2) in the above description.

Given any pair of states $\hat{\varrho}_1, \hat{\varrho}_0$, the **convex combination**

$$\hat{\varrho}_\lambda = \lambda\hat{\varrho}_1 + (1-\lambda)\hat{\varrho}_0\,, \qquad 0 \leq \lambda \leq 1 \tag{13.7}$$

is also a possible state, i.e. for every λ in the range $0 \leq \lambda \leq 1$ the operator $\hat{\varrho}_\lambda$ is selfadjoint, non-negative, normalized. Hence the set of states, denoted $\mathcal{S}(\mathcal{H})$ is a convex set.

Example The state operator of a qubit can always be written in the **Bloch vector representation**

$$\hat{\varrho} = \tfrac{1}{2}\left(\hat{1} + \vec{s}\cdot\hat{\vec{\sigma}}\right) \tag{13.8}$$

where the real vector $\vec{s} = (s_x, s_y, s_z) \in \mathbb{R}^3$, $|\vec{s}| \leq 1$, measures the *polarization* of the qubit,

$$s_i = \mathrm{tr}\{\hat{\sigma}_i \hat{\varrho}\} \equiv \langle \hat{\sigma}_i \rangle \,, \qquad i = x, y, z \,. \tag{13.9}$$

Simple again. Since $\hat{\sigma}_i$ are selfadjoint and s_i are real, the operator $\hat{\varrho}$ is self-adjoint. And since $\hat{\sigma}_i$ are traceless, $\hat{\varrho}$ is normalized. Finally, with $|\vec{s}| \leq 1$, positivity of $\hat{\varrho}$ is guaranteed (End-of-proof). As $0 \leq |\vec{s}| \leq 1$, states my be identified with points of a ball of radius one – the *Bloch sphere* Fig. 13.1. For pure states $|\vec{s}| = 1$, i.e. pure states are points on the surface the Bloch sphere with opposite points corresponding to orthogonal pure states $|\uparrow_s\rangle, |\downarrow_s\rangle$. Furthermore, with any two points $\vec{s}_1, \vec{s}_2 \in S^3$, the straight line joining these two points $\vec{s}(\lambda) = \lambda\vec{s}_1 + (1-\lambda)\vec{s}_2$, $0 \leq \lambda \leq 1$, lies entirely in the Bloch sphere. The Bloch sphere is a convex set indeed.

Generalizing for a quantum system with N-dimensional Hilbertspace, the state space $\mathcal{S}(\mathcal{H})$ may be parametrized by $N^2 - 1$ real parameters,[1] i.e. the set of states is a $N^2 - 1$ dimensional manifold *with boundary*. The boundary consists of all states which have at least one zero eigenvalue. Within the boundary manifold, pure states form a submanifold of dimension $2N - 2$.[2] As this is a set of measure zero of $\mathcal{S}(\mathcal{H})$, pure states are indeed the exception rather than the rule.

A particular form of a state operator, which is useful for general considerations, is provided by its **spectral representation**

$$\hat{\varrho} = \sum_{\nu=1}^{\dim\mathcal{H}} \varrho_\nu |\nu\rangle\langle\nu| \,, \tag{13.10}$$

[1] Hermitian = N^2 parameters, normalization = 1 condition

[2] $2N$ real numbers, minus 1 normalization, 1 global phase, which is unobservable

where $|\nu\rangle$ is an orthonormal basis of eigenvectors of $\hat{\varrho}$. Since $\hat{\varrho}$ is non-negative and normalized, the eigenvalues are restricted

$$\varrho_\nu \geq 0\,, \qquad \sum_{\nu=1}^{\dim\mathcal{H}} \varrho_\nu = 1\,. \tag{13.11}$$

In fact the νth eigenvalue ϱ_ν is the probability that in a maximal test, which is defined by the particular $|\nu\rangle$-basis, the νth channel will click, and the quantum system will be leave the device in state $|\nu\rangle$ as a result of that measurement.

The state operator of a **pure state** $|\psi\rangle$ is given by a rank-one projection $\hat{\varrho} = |\psi\rangle\langle\psi|$. A state operator which can not be written as a rank-one projection is called a **mixed state**. The state operator $\hat{\varrho} = \frac{1}{\dim\mathcal{H}}\mathrm{id}_\mathcal{H}$ is called a **complete mixture**, sometimes **chaotic state**.

Theorem: A state $\hat{\varrho} \in \mathcal{S}(\mathcal{H})$ is pure if and only if $\mathrm{Tr}\hat{\varrho}^2 = 1$.

The proof is elementary. Let $r_\mu, \mu = 1, \ldots, \dim\mathcal{H}$ denote the eigenvalues of $\hat{\varrho}$, and without loss of generality $r_\mu \leq r_1$. We have $\mathrm{Tr}\hat{\varrho}^2 = \sum_\mu r_\mu^2 \leq r_1 \sum_\mu r_\mu = r_1 \leq 1$. In the last inequality equality can only be realized if $r_1 = 1$, i.e. if $\hat{\varrho}$ is a pure state. *End-of-proof*

The spectral representation of Eq. (13.8) reads

$$\hat{\varrho} = \varrho_+ |\uparrow_s\rangle\langle\uparrow_s| + \varrho_- |\downarrow_s\rangle\langle\downarrow_s|\,, \tag{13.12}$$

where $|\uparrow_s\rangle, |\downarrow_s\rangle$ are the eigenstates of $\hat{\sigma}_s \equiv \vec{s}\cdot\hat{\vec{\sigma}}$, and

$$\varrho_\pm = \tfrac{1}{2}\left(1 \pm |\vec{s}|\right) \tag{13.13}$$

are the eigenvalues of $\hat{\varrho}$. The proof is simple. Recall, for $\vec{s} \in \mathbb{R}^3$, $\vec{s}\cdot\hat{\vec{\sigma}}|\uparrow_s\rangle = +|\vec{s}||\uparrow_s\rangle$, $\vec{s}\cdot\hat{\vec{\sigma}}|\downarrow_s\rangle = -|\vec{s}||\downarrow_s\rangle$. *End-of-proof*

The spectral representation may be viewed a particular convex decomposition – a decomposition into orthogonal pure states. For states of a qubit, with non-degenerate spectrum $\{\varrho_+, \varrho_-\}$, this decomposition is unique. Otherwise it is not: the complete mixture, for example, allows for infinitely many different decompositions.

To any preparation procedure there corresponds a unique state operator which completely specifies the (probabilistic) results of any conceivable test. However, the reverse is not true in general: it is impossible to infer the preparation procedure from the knowledge of the state operator. The passage of $|\uparrow_y\rangle$ through a SG device with orientation along the x-axis, for example, produces the same state as if the particle would pass through a SG device with the orientation along the z-axis: In both cases, the resulting state is a complete mixture, and nothing in the world could reveal whether the procedure 1 or the macroscopically different procedure 2 was used in preparing the particles.

Pure states are the extremal points of the convex set $\mathcal{S}(\mathcal{H})$. They can not be further decomposed in terms of other pure states. Yet in contrast to classical probability distributions, the decomposition of a quantum mechanical mixed state in terms of pure states is not unique. The complete mixture $\hat{\varrho} = \frac{1}{2}\hat{1}$, for example, admits different decompositions

$$\hat{\varrho} = \tfrac{1}{2}\left(|\uparrow_z\rangle\langle\uparrow_z| + |\downarrow_z\rangle\langle\downarrow_z|\right) \tag{13.14}$$

$$= \tfrac{1}{2}\left(|\uparrow_x\rangle\langle\uparrow_x| + |\downarrow_x\rangle\langle\downarrow_x|\right). \tag{13.15}$$

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13.2 The QM 2.0 Postulate of Observables

Postulate II (Observable Postulate) To any observable A of a quantum system corresponds a linear Operator $\hat{A} : \mathcal{H} \to \mathcal{H}$, selfadjoint $\hat{A}^\dagger = \hat{A}$, the eigenvalues of which correspond the possible measurement values encountered in measuring A. In state $\hat{\varrho}$, the statistical mean of the measured values of A is given by

$$\langle \hat{A} \rangle_\varrho = \mathrm{Tr}(\hat{A}\hat{\varrho})\,. \tag{13.16}$$

In the spectral representation of $\hat{A}$ we would write $\hat{A} = \sum_\mu a_\mu \hat{P}_\mu$ with $\hat{P}_\mu = |a_\mu\rangle\langle a_\mu|$ the rank-one projection onto the eigenstate $|a_\mu\rangle$ to eigenvalue a_μ. Equation (13.16) then rewrites

$$\langle \hat{A} \rangle_\varrho = \sum_\mu a_\mu p_\mu, \qquad p_\mu = \mathrm{Tr}\{\hat{P}_\mu \hat{\varrho}\} \tag{13.17}$$

i.e. the weighted sum of the measurement values, the weights given by the probability p_μ, that measurement value a_μ will indeed be encountered.

The observable postulate reveals another source of randomness which is the "preparation by measurement".

Example Consider for example a qubit in state $|\psi\rangle$ which is first send throuth a SGM with orientation $\vec{a}$ and then forwarded to a second measurement device which measures an observable $\hat{A}$. According to the rules of quantum mechanics, for particles which leave the first SGM through the upper channel before entering the second device, $\langle \hat{A} \rangle = \langle \uparrow_a | \hat{A} | \uparrow_a \rangle$, while for particles which leave through the lower channel , $\langle \hat{A} \rangle = \langle \downarrow_a | \hat{A} | \downarrow_a \rangle$. With $p_{\uparrow_a} = |\langle \uparrow_a | \psi \rangle|^2$ the probability for the occurence of the first event, and $p_{\downarrow_a} = |\langle \downarrow_a | \psi \rangle|^2$ the probability for the second event, the overall expectation value of the observable $\hat{A}$ is given

by

$$\langle \hat{A} \rangle = \sum_{\mu=\uparrow_a,\downarrow_a} p_\mu \langle \mu | \hat{A} | \mu \rangle \,. \tag{13.18}$$

Introducing the operator

$$\hat{\varrho} = \sum_{\mu=\uparrow_a,\downarrow_a} p_\mu |\mu\rangle\langle\mu| \,, \tag{13.19}$$

the expectation value (13.18) can also be written

$$\langle \hat{A} \rangle = \mathrm{tr}\{\hat{\varrho}\hat{A}\} \,. \tag{13.20}$$

Note that the operator in Eq. (13.1) describes the situation after the passage through the first measurement device, and before entering the second measurement device. It should now be clear, that any measurement can well be viewed a preparation device. The concatenation of the preparation of $|\psi\rangle$ with the subsequent SGM (with orientation $\vec{a}$) is a preparation device which defines a quantum ensemble which can subsequently be measured by any observable A.

13.3 The QM 2.0 Postulate of Change

In the sequence "input–something happens–output" the "something happens" may quite generally called a **channel**. Channels provides the unifying framework for the description of all sorts of processes in quantum mechanics. Examples are (i) the reversible gate, i.e. ordinary Schrödinger dynamics, (ii) the spontaneous decay, other irreversible processes, and (iii) measurements.

The quantum mechanical notion of a channel is formulated

$$\hat{\varrho} \mapsto \hat{\varrho}' = \Phi(\hat{\varrho}) \,, \tag{13.21}$$

where Φ is a map on $\mathcal{S}(\mathcal{H})$, which maps the input to the output. Clearly, in order to map state operators onto state operators, this map (i) must respect self-adjointness, i.e. $\Phi(\varrho) = [\Phi(\varrho)]^\dagger$, it must be (ii) **positive** (map positive operators onto positive operators), and (iii) trace preserving.

In fact, the map Φ should be **completely positive** in order to qualify for a physical channel, that is if the system S under consideration is augmented by some auxiliary system R, sometimes called **ancilla** (the ancilla may be a qubit or just the rest of the universe), the map $\Phi^S \otimes I^R$, with Φ^S positive, and I^R the identity on R-StateSpace, should be positive on $\mathcal{S}(\mathcal{H}^{S\&R})$. If this is the case, the map Φ^S is called completely positive. For some positive, but not completely positive map, see the next section on the transposition (which is positive, but not completely positive). For the time being we explore positive maps.

Consider a quantum mechanical system which was prepared in some state $\hat{\varrho}$, say. In a **maximal test**, which is specified by a orthonormal basis $\{|w_\mu\rangle | \mu = 1, 2, \ldots, \dim\mathcal{H}\}$, the probability for any system drawn from the ensemble $\hat{\varrho}$ to end up in the μth channel is given by (recall Sec. 6.2)

$$p(\mu|\hat{\varrho}) \equiv \langle w_\mu|\hat{\varrho}|w_\mu\rangle = \mathrm{Tr}\hat{\varrho}\hat{P}_\mu \tag{13.22}$$

where $\hat{P}_\mu = |w_\mu\rangle\langle w_\mu|$ is a rank-one projection.

The **measurement** is a *physical process*. Hence it can be described as a map on state space, Eq. (13.21). In the particular case of a maximal test, if none of the systems are filtered according to their channel (non-selective measurement), this map is given by

$$\Phi(\hat{\varrho}) = \sum_{\mu=1}^{\dim\mathcal{H}} p(\mu|\hat{\varrho})\hat{P}_\mu \,. \tag{13.23}$$

If those systems are filtered which passed the μth channel (selective measurement),

say, the map reads

$$\Phi(\hat{\varrho}) := \frac{\hat{P}_\mu \hat{\varrho} \hat{P}_\mu}{\mathrm{Tr} \hat{P}_\mu \hat{\varrho} \hat{P}_\mu} \equiv |w_\mu\rangle\langle w_\mu| \,, \tag{13.24}$$

where the second equation is characteristic for the selection in a maximal test. Note, that a selective measurement map is nonlinear on state space, while the non-selective is linear. Also note that both, the selective and non-selective measurements may well be viewed a preparation procedure.

The concept of a measurement can be generalized to allow not only for maximal tests but for all other sorts of possible measurements. Quite generally, a measurement is specified by a family $\{\hat{K}_i | i = 1, 2, \ldots, N\}$ of **measurement operators**, the only restriction being that they must obey the completeness relation,

$$\sum_{i=1}^{N} \hat{K}_i^\dagger \hat{K}_i = \hat{1} \,. \tag{13.25}$$

Here the index i refers to the measurement outcome, and for $\hat{\varrho}$ the state of the system immediately before the measurement, the probability that the result i occurs is given by

$$p(i|\hat{\varrho}) := \mathrm{Tr} \hat{K}_i \hat{\varrho} \hat{K}_i^\dagger \,. \tag{13.26}$$

The selective version of the measurement is described by the map

$$\hat{\varrho} \to \Phi(\hat{\varrho}) = \frac{\hat{K}_i \varrho \hat{K}_i^\dagger}{\mathrm{Tr} \hat{K}_i \varrho \hat{K}_i^\dagger} \,, \tag{13.27}$$

and the non-selective measurement is described by

$$\hat{\varrho} \to \Phi(\hat{\varrho}) = \sum_{i=1}^{N} \hat{K}_i \varrho \hat{K}_i^\dagger \,. \tag{13.28}$$

The measurement is called **projection valued measurement (PVM)** if the $\hat{K}_i^\dagger \hat{K}_i$ form a complete set of projections. Otherwise the set $\{\hat{K}_i^\dagger \hat{K}_i | i = 1, \ldots, N\}$ is called a **positive operator valued measure (POVM)**. The maximal test, finally, is a PVM with rank-one projections.

Consider a spin-1 particle, the eigenstates of $\hat{S}_z/\hbar$ denoted $|-1\rangle$, $|0\rangle$, $|1\rangle$ to eigenvalues $m = -1, 0, 1$, respectively. Consider the projections $\hat{P}_1 = \frac{1}{\sqrt{2}}(|1\rangle\langle 1| + |0\rangle\langle 0|)$, $P_2 = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |-1\rangle\langle -1|)$, $\hat{P}_3 = \frac{1}{\sqrt{2}}(|-1\rangle\langle -1| + |1\rangle\langle 1|)$.

13.4 Qubit Liouville-von Neumann equation

For the quantum mechanical state operator, the Schrödinger equation reads

$$\frac{d}{dt}\hat{\varrho} = -\frac{i}{\hbar}\left[\hat{H}, \hat{\varrho}\right], \tag{13.29}$$

with the solution being given by

$$\hat{\varrho}_f = \hat{U}\,\hat{\varrho}_i \hat{U}^\dagger, \tag{13.30}$$

where $\hat{U}$ is the unitary operator ().

The Schrödinger equation is the quantum analog of the Liouville equation of classical *Hamiltonian* mechanics. And as Hamiltonian mechanics can not account for damping and other irreversible phenomena, the Schrödinger equation (13.29) can not account for irreversibility either.

Consider, for example, an atom which was prepared at time $t = 0$ in its excited state $|e\rangle$. After sufficiently long a time, one will find, with high probability, the atom in its ground state $|g\rangle$: while waiting, the atom has spontaneously – i.e. without

intervention – emitted a photon, and thereby made a transition into its ground state. If you insist, you may describe this process within the framework of the Schrödinger equation, but in this case you would be forced to consider not only the dynamics of the atom, but the temporal evolution of a combined system "atom+electromagnetic field". Yet not being interested in the field degrees of freedom, but only in the atom, there may be a dynamical equation for the atom state operator alone, which would bypass the atom Schrödinger equation (13.29).

The equation in question, which will be derived from first principles in a subsequent lecture (spin-Boson model), reads

$$\frac{d}{dt}\hat{\varrho} = -\frac{i}{\hbar}\left[\hat{H}_{\text{eff}}\hat{\varrho} - \hat{\varrho}\hat{H}_{\text{eff}}^{\dagger}\right] + \gamma\hat{\sigma}_{-}\hat{\varrho}\hat{\sigma}_{+}\,, \tag{13.31}$$

where $\hat{\sigma}_{+} = \hat{\sigma}_{-}^{\dagger} = |e\rangle\langle g|$, and *effective Hamiltonian*

$$\hat{H}_{\text{eff}} = \hbar\left(\omega_0 - i\frac{\gamma}{2}\right)\hat{\sigma}_{+}\hat{\sigma}_{-}\,. \tag{13.32}$$

Here, the parameters ω_0 and γ refer to the Bohr transition frequency and the rate of spontaneous emission (natural linewidth), respectively. The inverse $1/\gamma$ is also called natural life time (of the excited state $|e\rangle$).

Both, the pure Schrödinger equation and the equation (13.31) can be written in the form of a **Liouville-von Neumann equation**

$$\frac{d}{dt}\hat{\varrho} = \mathcal{L}\,\hat{\varrho} \tag{13.33}$$

where the action of the **super operator** $\mathcal{L}$ is defined by the Eqs (13.29) or (13.31), respectively.

The solution of the Liouville-von Neumann equation provides a map on state space, $\hat{\varrho}_0 \mapsto \hat{\varrho}_t = e^{\mathcal{L}t}\hat{\varrho}_0$. In order to qualify for a possible physical process, this map

must necessarily be (i) positive, and (ii) trace preserving. The pure Schrödinger dynamics, $e^{\mathcal{L}t}\hat{\varrho}_0 = \hat{U}\hat{\varrho}_0\hat{U}^\dagger$, certainly obeys these requirements. But what about the Liouville-von Neumann equation of spontaneous emission, Eq. (13.31)?

Recall that, as any state operator of a qubit may be parametrized in terms of its Bloch vector – cf Eq. (13.8) – the dynamics of the state operator may be described in terms of the dynamics of the Bloch vector. With the convention $\hat{\sigma}_\pm = (\hat{\sigma}_x \pm i\hat{\sigma}_y)/2$, the corresponding Bloch equations read

$$\dot{s_x} = -\frac{\gamma}{2}s_x - \omega_0 s_y\,, \tag{13.34}$$

$$\dot{s_y} = -\frac{\gamma}{2}s_y + \omega_0 s_x\,, \tag{13.35}$$

$$\dot{s_z} = -\gamma s_z - \gamma\,. \tag{13.36}$$

The solution of these equations are found

$$s_x(t) = s_x(0)e^{-\gamma t/2}\cos\omega_0 t - s_y(0)e^{-\gamma t/2}\sin\omega_0 t\,, \tag{13.37}$$

$$s_y(t) = s_y(0)e^{-\gamma t/2}\cos\omega_0 t + s_x(0)e^{-\gamma t/2}\sin\omega_0 t\,, \tag{13.38}$$

$$s_z(t) = s_z(0)e^{-\gamma t} - 1 - e^{-\gamma t}\,. \tag{13.39}$$

Since for $\vec{s}(0)$ real, $\vec{s}(t)$ is real as well, $\hat{\varrho}$ remains self-adjoint, and since, for $t \geq 0$, $|\vec{s}(t)| < |\vec{s}(0)|$, $\hat{\varrho}(t)$ remains positive. Finally, since $\mathrm{Tr}\{\mathcal{L}\hat{\varrho}\} = 0$, and thus $\frac{d}{dt}\mathrm{Tr}\hat{\varrho} = 0$, $\hat{\varrho}$ remains normalized. Hence indeed – the Liouville-von Neumann propagator $e^{\mathcal{L}t}$ is a positive trace preserving map on state space.

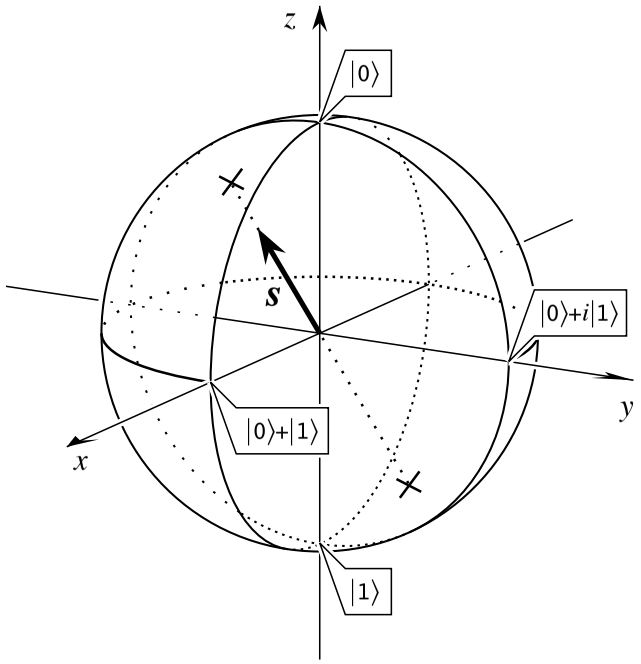

Figure 13.1: The Bloch sphere of a qubit. Pure states correspond to points on the surface, mixed states correspond to point in the interior.

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