Quantum Information (WiSe 2019/2020)

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Problem Set No 3 ($20 + \pi$ scores)[1]

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▷ Aufgabe 1 (Decrypt this!) (2 scores)

As an inofficial employee of your countrie's special service you are of course well aware of the relative frequencies of the letters in English language documents (see Fig 1 for a reminder). Hence for you it is a piece of cake to decrypt the following cryptogram

```
,(flphjmkcfj,kslg-zdsfclzplvzcchj8vk,8zjl8
il,(k,lzpl-fg-zmhv8jalk,lzjflgz8j,lf8,(f-l
fnkv,selz-lkgg-zn8ck,fselklcfiikaflifsfv,f
mlk,lkjz,(f-lgz8j,qluvskhmfli(kjjzj.lwrt4o
```

which is known to be the result of a simple substitution cipher executed on an English text. What message did the sender of this cryptogram try to conceal?

Letter	Percentage	Letter	Percentage	Letter	Percentage	Letter	Percentage
a	5.75	h	3.13	o	6.89	v	0.69
b	1.28	i	5.99	p	1.92	w	1.19
c	2.63	j	0.06	q	0.08	x	0.73
d	2.85	k	0.84	r	5.08	y	1.64
e	9.13	l	3.35	s	5.67	z	0.07
f	1.73	m	2.35	t	7.06	-	19.28
g	1.33	n	5.96	u	3.34		

Abbildung 1: Distribution (percentage) over the 27 outcomes for a randomly selected letter in an English language document *The Frequently Asked Questions Manual for Linux*. Quoted after: David J.C. MacKay, Cambridge, UK.

▷ Aufgabe 2 (Infinite Library) (2 Scores)

In preparing for the move to his "Auslandsemester" your friend plans to take his book collection all along. His idea is to code the books into a real number $\theta \in [0, \pi]$ which determines, via $|\psi\rangle = \cos(\vartheta/2)|\uparrow_z\rangle + \sin(\vartheta/2)|\downarrow_z\rangle$, the state of a qubit. "With only one qubit in my pocket" he argues "there will be no problems with luggage". Upon arrival he plans to read out the state, and decode ϑ in order to enjoy his book collection. Not being an expert in Quantum information, he asks you whether this will work. You answer

- ☐ No
- ☐ Yes
- ☐ Possibly

[1]Problems with transcendental scores are facultative nuts. Nuts have high nutrition value ...

▷ **Aufgabe 3 (Qubitology)** (7 scores)

Recall that for the qubit the Pauli spin operator $\hat{\vec{\sigma}} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$ is of paramount importance. Its cartesian components obey angular momentum commutation relations

$$[\hat{\sigma}_x, \hat{\sigma}_y] = 2i\hat{\sigma}_z\,, \qquad \text{with } xyz \text{ cyclic}\,, \tag{1}$$

and the specific spin-$\frac{1}{2}$ anti-commutation relations

$$\{\hat{\sigma}_i, \hat{\sigma}_j\} = 2\delta_{ij}\hat{1}\,, \qquad i,j = x,y,z\,. \tag{2}$$

where $\hat{1}$ is the identity operator on $\mathcal{H}_{\text{qubit}} \simeq \mathbb{C}^2$, which we shall occasionally denote $\hat{\sigma}_0 \equiv \hat{1}$.

Acting in a two-dimensional Hilbert space, the Pauli operators admit a representation in terms of hermitian 2×2 matrices. Adopting the convention

$$|\uparrow_z\rangle \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad |\downarrow_z\rangle \mapsto \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{3}$$

for the eigenvectors of $\hat{\sigma}_z$, the standard matrix representation reads

$$\hat{\sigma}_0 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \hat{\sigma}_x \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \hat{\sigma}_y \mapsto \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \hat{\sigma}_z \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{4}$$

Matrix representations are quite useful – their two-dimensional layout is better adapted to human parallel processing than the serial layout of the abstract notation ...

(a) Let $\vec{a}$ be a (real) spatial unit vector, $|\vec{a}| = 1$, and denote

$$\hat{\sigma}_a := \vec{a}\cdot\hat{\vec{\sigma}} \tag{5}$$

the cartesian component of $\hat{\vec{\sigma}}$ in $\vec{a}$-direction. Confirm that $\hat{\sigma}_a$ is self adjoint. Furthermore

$$\text{tr}\hat{\sigma}_a = 0\,, \tag{6}$$

and

$$\hat{\sigma}_a^2 = \hat{1}\,. \tag{7}$$

Conclude that the spectrum of $\hat{\sigma}_a$ is given $\{+1, -1\}$.

(b) Confirm the master identity

$$\exp\{i\alpha\hat{\sigma}_a\} = \cos(\alpha)\hat{1} + i\sin(\alpha)\hat{\sigma}_a\,. \tag{8}$$

(c) Denote $|\uparrow_a\rangle, |\downarrow_a\rangle$ the eigenvectors of $\hat{\sigma}_a$. Confirm the spectral representation

$$\hat{\sigma}_a = |\uparrow_a\rangle\langle\uparrow_a| - |\downarrow_a\rangle\langle\downarrow_a|\,. \tag{9}$$

Express $|\uparrow_a\rangle, |\downarrow_a\rangle$ in terms of a linear combination of $|\uparrow_z\rangle, |\downarrow_z\rangle$.

(d) Let $\vec{a}$ and $\vec{b}$ denote spatial vectors, not necessarily unit vectors. Confirm the beautiful identiy

$$\left(\vec{a}\cdot\hat{\vec{\sigma}}\right)\left(\vec{b}\cdot\hat{\vec{\sigma}}\right) = (\vec{a}\cdot\vec{b})\hat{1} + i(\vec{a}\times\vec{b})\cdot\hat{\vec{\sigma}}\,. \tag{10}$$

and conclude

$$\left[\vec{a}\cdot\hat{\vec{\sigma}}, \vec{b}\cdot\hat{\vec{\sigma}}\right] = 2i(\vec{a}\times\vec{b})\cdot\hat{\vec{\sigma}}\,. \tag{11}$$

(e) For unit vectors $\vec{a}, \vec{b}$ confirm

$$\langle\uparrow_a|\hat{\sigma}_b|\uparrow_a\rangle = \vec{a}\cdot\vec{b}\,, \tag{12}$$

and thus obtain the *transition probabilities*

$$p(\uparrow_b\mid\uparrow_a) \equiv |\langle\uparrow_b\mid\uparrow_a\rangle|^2 = \tfrac{1}{2}\Big(1+\vec{a}\cdot\vec{b}\Big)\,, \tag{13}$$

$$p(\downarrow_b\mid\uparrow_a) \equiv |\langle\downarrow_b\mid\uparrow_a\rangle|^2 = \tfrac{1}{2}\Big(1-\vec{a}\cdot\vec{b}\Big)\,. \tag{14}$$

(f) For many applications the ladder operators are useful. The descending operator, also called annihilation operator, is defined

$$\hat{\sigma} := \frac{1}{2}\left(\hat{\sigma}_x - i\hat{\sigma}_y\right)\,, \tag{15}$$

and thus the ascending operator, also called creation operator, is given

$$\hat{\sigma}^\dagger = \frac{1}{2}\left(\hat{\sigma}_x + i\hat{\sigma}_y\right)\,. \tag{16}$$

Confirm the matrix representation

$$\hat{\sigma} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\,, \qquad \hat{\sigma}^\dagger = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\,. \tag{17}$$

(g) Express $\hat{\sigma}_x$ and $\hat{\sigma}_y$ in terms of $\hat{\sigma}$ and $\hat{\sigma}^\dagger$, confirm the algebra

$$\hat{\sigma}^2 = \hat{\sigma}^{\dagger 2} = 0\,, \qquad \left\{\hat{\sigma}, \hat{\sigma}^\dagger\right\} = \hat{1}\,, \qquad \left[\hat{\sigma}, \hat{\sigma}^\dagger\right] = -\hat{\sigma}_z\,, \tag{18}$$

and admire the reduction in complexity: instead of using 4 different Pauli matrices, you evidently need only one (and its adjoint) in order to talk about spin-1/2 particles.

(h) Confirm the identity

$$\exp\{i\omega t\hat{\sigma}_z\}\hat{\sigma}\exp\{-i\omega t\hat{\sigma}_z\} = e^{-i\omega t}\hat{\sigma} \tag{19}$$

and alternative formulation

$$\exp\{i\omega t\hat{\sigma}^\dagger\hat{\sigma}\}\hat{\sigma}\exp\{-i\omega t\hat{\sigma}^\dagger\hat{\sigma}\} = e^{-i\omega t}\hat{\sigma} \tag{20}$$

Enjoy the formal similarity to the harmonic oscillator solution

$$\exp\{i\omega t\hat{a}^\dagger\hat{a}\}\hat{a}\exp\{-i\omega t\hat{a}^\dagger\hat{a}\} = e^{-i\omega t}\hat{a} \tag{21}$$

▷ **Aufgabe 4 (Qubit Communication)** (3 scores)

Alice prepares a qubit either in state $|1\rangle = |\uparrow_a\rangle$, or in state $|2\rangle = |\downarrow_a\rangle$. The probabilities that she does the one or the other are given by $p_1 = p_2 = 0.5$.

Although Bob knows Alice's choice $\vec{a}$, he measures with orientation $\vec{b}$ (just to see how much he could learn); the two outcomes are labeled $+$ and $-$.

(a) What is the probability $P_{\sigma|i}$ that Bob finds σ, $\sigma = \pm$, given that Alice transmitted $|i\rangle$, $i = 1, 2$?

(b) What is the probability that Bob finds $\sigma = \pm$ irrespective of what Alice transmitted?

(c) What is the probability that Alice in fact transmitted $|i\rangle$ given that Bob found σ?

(d) What orientation should Bob chose for perfect communication? How many bits per qubit are revealed with this orientation?

Background This example proves that in an ideal world, where there is no disturbing influence from the enviroment, 100-percent reliable communication is possible using qubits. The only point sender and receiver must be aware of is to chose equal alignment of their Stern-Gerlach devices. The example also indicates that in this particular context, the maximal amount of information which a single qubit can carry is one bit. Later in the lecture we shall see that in a different context, which involves entanglement, one can cram two bits in a single qubit.

▷ **Aufgabe 5 (Daimler-Benz)** (6 scores)

Alice prepares a qubit in the up-state $|\uparrow_i\rangle$ with respect to one out of three possible quantization axis $\vec{a}_i$, $i = 1, 2, 3$, where the $\vec{a}_i$ form a co-planar “Mercedes-Stern”,

$$\sum_{i=1}^{3} \vec{a}_i = 0\,. \tag{22}$$

Bob knows the possible directions $\vec{a}_i$, but he does not know which particular direction Alice has chosen. What is his initial level of ignorance? How much could he expect to learn about Alice’s choice, and what is his optimal strategy?

▷ **Aufgabe 6 (Monty Hall)** (π scores)

Participating in a game show you have luckily reached the final round where you are given the opportunity to collect your prize. The prize is hidden behind one of three doors, all of which are closed. You are asked to point at that door behind which you expect the prize is hidden. At that point, the door is not yet opened, but Monty Hall, the show master (who knows where the prize is hidden), opens another door instead, behind which there is no prize. He then asks you whether you insist on your initial choice or you rather prefer to switch to the remaining door. Once you have announced your decision, the corresponding door is opened, and you may take home whatever is behind that door (i.e. the prize, if you are lucky, or nothing, if you are unlucky). How would you decide?

Background: The problem, which also runs under the name “goat problem” (the prize is a goat), became famous when in the eighties, Marilyn von Savant presented the correct solution in the Scientific American. Her solution was fiercly attacked by even the most prestigeous experts in statistical analysis, who – in the end – were all wrong and Marilyn was all right. Meanwhile, the game has been quantized – see *T*he Quantum Monty Hall Problem by G. M. D’Ariano, R. D. Gill, M. Keyl, B. Kümmerer, H. Maassen, and R. F. Werner, Quant. Inf. Comp. **2** (2002), 355.