Theoretical Physics V – Quantum Mechanics II

Winter Semester 2019/20
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Problem Set #7 Hand Out: 14 January 2020
Hand In: 28 January 2020

Problem 7.1 – Nuclei and Neutron Stars (10 points)

We seek to describe a bound state of many nucleons (i.e., neutrons or protons) that are confined in a potential that they create on their own. In nuclear physics, this potential can be modelled from the Yukawa interaction between two nucleons. In astrophysics, a very large collection of nuclei is called a neutron star: here the binding force is simply gravity.

The nuclei are bound in a potential $\Phi(r)$ that we may assume to have spherical symmetry, neglecting the rotation of the nucleus (star). In addition, the nucleons are "cold" so that all states up to the Fermi energy ϵ_F are occupied, the others are empty. Our final assumption is to treat the nucleons in the non-relativistic approximation.

(a) Check that the non-relativistic approximation is consistent: if the nucleus has a size (radius) R and the potential is approximated by a spherical box, then the lowest momentum for a nucleon confined in the potential is $\ldots p \sim \hbar/R$ (why?). A nucleon of mass M then behaves non-relativistically if $cp \ll Mc^2$ (why?). Compare the two sides in this inequality. It may be useful to know that $\hbar c \approx 197\,\mathrm{MHz\,fm}$.

(b) Justify that the potential solves the generalised Poisson equation

$$\nabla^2\Phi - \kappa^2\Phi \,=\, g\rho$$

where ρ is the nucleon density. For the nucleus, $1/\kappa = \hbar/(m_\pi c)$ is related to the pion mass (compute the length $1/\kappa$) and the coupling constant g specifies the strength of the nucleon-nucleon interaction. For the neutron star, $\kappa = 0$ (why?) and $g = 4\pi GM$ where G is Newton's constant.

(c) In the Thomas-Fermi model, the local density ρ of nucleons is given by the density of a degenerate Fermi gas:

$$\rho = \frac{N_s}{6\pi^2}k_F^3\,, \qquad \frac{\hbar^2 k_F^2}{2M} = \epsilon_F - \Phi \tag{7.1}$$

Justify this value for the Fermi momentum k_F and $N_s = 2$ (4) for the neutron star (the nucleus). Show that the generalised Poisson equation yields

$$\nabla^2\Phi - \kappa^2\Phi \,=\, K\,(\epsilon_F - \Phi)^{3/2}\,, \quad \text{with} \quad K \,=\, \frac{8\sqrt{2}GM^{7/2}}{3\pi\hbar^3}\,.$$

where the constant K is given for the neutron star.

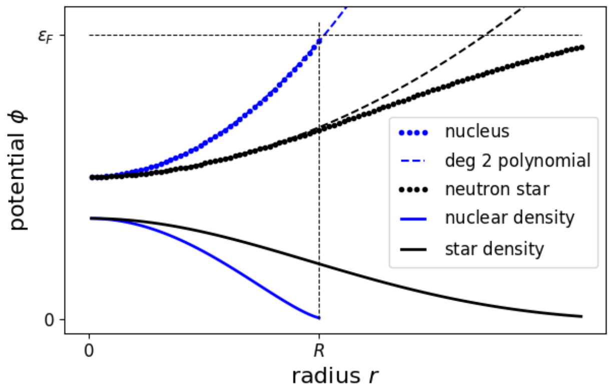
(d) Shift the zero of the potential to a suitable value, make the substitution $u(r) = -r\Phi(r)$, and find the differential equation

$$u''(r) - \kappa^2 u \;=\; -K\, r^{-1/2}\, u^{3/2}(r)\,.$$

(e) This *Lane-Emden* differential equation does not have an analytical solution. Since u must be odd, we make the Ansatz

$$u(r) \;=\; \phi_0 r + \alpha r^3 + \beta r^5 + \dots\,,$$

and apply the brutal approximation of keeping only the first two terms. A comparison to a numerical solution is shown in the Figure.



For the neutron star, the problem has only one free parameter, for example the depth of the potential. Compute the dependent parameter α from the asymptotic form near $r = 0$. Compute the radius R of the neutron star and its mass M in units of the solar mass.

(f) In the shell theory of nuclear structure, one often takes the simple potential $\Phi(r) = kr^2$. Construct the density profile $\rho = \rho(r)$ from Eq.(7.1) and compute the number A of nucleons and the nuclear radius R as a function of the chemical potential ϵ_F.

Problem 7.2 – Lagrangians for Fields in the Standard Model (10 points)

If you want to find a "World formula", then your starting point is the Lagrangian density (in short: Lagrangian) for the fields that represent the particles in the World. There are essentially three different forms that are relevant, spin-0 and spin-1 bosonic fields and a spin-1/2 fermionic field.

(a) A spin-0 bosonic field, for example the Higgs boson, can be described by

$$\mathscr{L} = \tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi) \tag{7.2}$$

Justify that $\mathscr{L}$ is a Lorentz scalar. Since the integral over $\mathscr{L}$ is the dimensionless action (if we put $\hbar = c = 1$), which physical unit has the field ϕ? (Note that masses, energies, momenta, and inverse lengths all have the same unit.) Construct the potential $V(\phi)$ in such a way that the Euler-Lagrange equation for ϕ corresponds to the wave equation for a relativistic particle of mass m. (*Answer*: $V(\phi) = \frac{1}{2}m^2\phi^2$.)

(a′) Show that in the non-relativistic limit where $\phi(t, \mathbf{x}) \approx \mathrm{e}^{-\mathrm{i}mt}\psi(t, \mathbf{x}) + \text{c.c.}$, one gets the Schrödinger equation. (You can start from the Lagrangian or from the wave equation.)

(a″) If the scalar boson is charged, then one works with a complex field ϕ and the covariant derivative $D_\mu = \partial_\mu - \mathrm{i}eA_\mu$. The Lagrangian becomes

$$\mathscr{L} = D_\mu\phi^*\,D^\mu\phi - V(\phi)$$

(Raising and lowering of indices is done with the Minkowski tensor, $D^\mu = g^{\mu\nu}D_\nu$. Einstein convention = double indices are summed over.) For the Higgs boson, the potential is often taken in the form $V(\phi) = -m^2|\phi|^2 + \lambda|\phi|^4$. What is the dimension of λ? Why is m *not* the mass of the Higgs boson? What values for ϕ minimise the potential?

(b) Spin-1 bosons like the photon or gluons are described by vector potentials, for example A_μ. Show that the Maxwell equations follow from the following action

$$\mathscr{L} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - j^\mu A_\mu \tag{7.3}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor and j^μ is a current density that does not depend on A_μ. (Don't be afraid by missing factors ε_0 or μ_0.) What are the natural dimensions of A_μ and the charge e in the covariant derivative? What happens if you add the term $m^2A^\mu A_\mu$ to the Lagrangian? Analyse the equation of motion (*Jargon*: "the photon becomes massive") and the gauge invariance of the Lagrangian ("gauge invariance is broken").

(c) Fermion fields are described by Dirac spinors Ψ with the Lagrangian

$$\mathscr{L} = \overline{\Psi}\gamma^\mu\mathrm{i}D_\mu\Psi - \Lambda\overline{\Psi}\phi\Psi$$

where $\overline{\Psi}$ is the adjoint spinor. Work out the Euler-Lagrange equation of motion for Ψ: what is the role played by the second term? (*Jargon*: "the fermion gets its mass from the coupling to the scalar field.")

(d) Several fields that interact are constructed by adding their Lagrangians. Interactions can be recognised by looking for product terms with three or more fields in the total Lagrangian. Make a list of the interactions in the above examples (charged Higgs boson, electromagnetic field, charged fermion). Current densities j^μ are typically quadratic in the fields – compute their forms in the above examples and check their physical units.

(e) These constructions are generalised in the Standard Model to so-called non-Abelian gauge theories. Here, the covariant derivative is enlarged to the form

$$D_\mu = \partial_\mu - \mathrm{i}g \sum_a \tau_a B^a_\mu \tag{7.4}$$

where the τ_a are hermitean matrices that are the generators of the symmetry group; to each of these corresponds a vector potential B^a_μ. Dimension of g? The definition of the field tensor is generalised to $F_{\mu\nu} = [D_\mu, D_\nu]$. One usually denotes the commutator $[\tau_a, \tau_b] = \mathrm{i} f^c_{ab} \tau_c$, calling the f^c_{ab} the "structure constants" of the symmetry group. Show that $F_{\mu\nu}$ contains a sum over terms linear (like $\partial_\mu B^a_\nu - \partial_\nu B^a_\mu$) and quadratic in the B^a_μ. Take the first term of Eq.(7.3) as Lagrangian and explain the jargon: "Non-abelian gauge bosons couple among themselves."