Einführung in die Quantenoptik I

Wintersemester 2019/20
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Problem 2.1 – Resonance approximation and *rotating frame* (7 points)

The time-dependent state for a two-level atom (with energies E_e and E_g) can be specified by the amplitudes $c_e(t)$ and $c_g(t)$. In the lecture, we applied the transformation

$$|\psi(t)\rangle = U_0(t)|\psi'(t)\rangle \tag{2.1}$$

to the "interaction picture". (1) Show that in this picture, the amplitudes $c'_e(t)$ and $c'_g(t)$ are related by

$$c_e(t) = c'_e(t)\,\mathrm{e}^{-\mathrm{i}(E_g/\hbar+\omega)t} \tag{2.2}$$

$$c_g(t) = c'_g(t)\,\mathrm{e}^{-\mathrm{i}E_g t/\hbar} \tag{2.3}$$

to the "old" ones. Which value for ω did we use to transform into the interaction picture?

(2) Another choice in Eq.(2.2) is $\omega = \omega_L$ (the "*rotating frame*"), where ω_L is the (carrier) frequency of the laser field. Show that the time-dependent Schrödinger equation in this picture looks like

$$\mathrm{i}\hbar\partial_t c'_e(t) = -\hbar\Delta c'_e(t) - \mathbf{d}_{eg}\cdot\mathbf{E}(t)\,\mathrm{e}^{\mathrm{i}\omega_L t}c'_g(t) \tag{2.4}$$

(similar equation for $c'_g(t)$) where $\Delta = \omega_L - \omega_A$ is the "detuning" and $\mathbf{d}_{eg} = \langle e|\mathbf{d}|g\rangle$ is the "transition dipole".

(3) Show that within the resonance approximation (also called *rotating wave approximation*), the atom-light coupling is simplified as follows:

$$\mathbf{d}_{ge}\cdot\mathbf{E}(t)\,\mathrm{e}^{\mathrm{i}\omega_L t} \approx \mathbf{d}_{ge}\cdot\boldsymbol{\mathcal{E}}(t) \tag{2.5}$$

where the complex laser amplitude (or "envelope") $\boldsymbol{\mathcal{E}}(t)$ would be constant for a cw laser. Sketch in words the advantages of this approximation.

Problem 2.2 – Exponentiating matrices (7 points)

Within suitable approximations, the time-dependent Schrödinger equation for a two-level system (amplitudes c_e and c_g) can be written in the form

$$\begin{aligned} \mathrm{i}\partial_t c_e &= -\frac{\Delta}{2} c_e + \frac{\Omega}{2} c_g && (2.6) \\ \mathrm{i}\partial_t c_g &= \frac{\Omega^*}{2} c_e + \frac{\Delta}{2} c_g && (2.7) \end{aligned}$$

where Δ and Ω are two parameters with the units of frequency. [5 Bonus points: show that with a suitable choice of the phase difference between c_e and c_g, you can "transform away" the phase of the complex number Ω. We may therefore assume without loss of generality that Ω is real in the following.]

(1) Show that the "effective Hamiltonian" of this Schrödinger equation is a hermitian 2×2-matrix of trace zero. Calculate its eigenvalues: $\pm\varpi/2$ with $\varpi = \sqrt{\Delta^2 + \Omega^2}$.

(2) Show that the time evolution operator is given by

$$\begin{aligned} U(t) &= \cos(\varpi t/2)\mathbb{1} + \mathrm{i}\sigma \sin(\varpi t/2) && (2.8) \\ \sigma &= \frac{1}{\varpi} \begin{pmatrix} -\Delta & \Omega \\ \Omega & \Delta \end{pmatrix} && (2.9) \end{aligned}$$

Problem 2.3 – Schrödinger equation and entropy (6 points)

In the lecture, we are going to learn about two descriptions for the dynamics of a two-level atom: rate equations and the Schrödinger equation. They have a fundamental difference: in one case, the entropy of a two-level system is invariant.

The quantum-mechanical entropy is introduced since John von Neumann by the following formula:

$$S = -\sum_a p_a \log p_a \qquad (2.10)$$

where the p_a are the eigenvalues of the density operator ρ, and $\log$ the natural logarithm. We set that $\lim_{p\to 0} p \log p = 0$ (consistent with de l'Hôpital's, rule, Python command: `scipy.special.xlogy`). If the state is given in terms of a ket $|\psi\rangle$, then

$$\rho = |\psi\rangle\langle\psi| \qquad (2.11)$$

(1) Show that $S(|\psi\rangle) = 0$. The solution of the time-dependent Schrödinger equation is $|\psi(t)\rangle = U(t)|\psi(0)\rangle$. Show that $|\psi(t)\rangle$ and $|\psi(0)\rangle$ have the same entropy.

(2) For a two-level system the density operator is generally a hermitian 2×2 matrix of the form

$$\rho = \begin{pmatrix} p_e & \rho_{eg} \\ \rho_{ge} & p_g \end{pmatrix} \qquad (2.12)$$

where p_e, p_g are the probabilities to find the system in the states $|e\rangle$, $|g\rangle$. Assume for simplicity that the non-diagonal elements $\rho_{eg} = \rho_{ge} = 0$. In that case, rate equations can be applied. Consider the simple example

$$\partial_t p_e = -\Gamma p_e + \gamma p_g \tag{2.13}$$

$$\partial_t p_g = +\Gamma p_e - \gamma p_g \tag{2.14}$$

and show that the entropy $S(\rho)$ becomes time-dependent. Which stationary value for S do you get in the special case $\gamma = \Gamma$?

There is no need to solve the rate equations. If you are interested, however, an *Ansatz* with exponential functions yields rapidly a solution.