Einführung in die Quantenoptik I

Wintersemester 2019/20
Carsten Henkel

Problem Set #5 Hand out: 03 Dec 2019
Hand in: 17 Dec 2019

Problem 5.1 – Thermal radiation and Planck spectrum (10 points)

The formula found in 1900 by Max Planck for the spectrum of a "black body" (i.e., radiation in thermal equilibrium at temperature T) marks the birth of quantum physics.

(1) Show that in classical electromagnetism, the number of modes per unit volume and per unit frequency interval, $\rho(\omega)$ is given by

$$\rho(\omega)\frac{\mathrm{d}\omega}{2\pi} = \frac{8\pi}{(2\pi)^3}\frac{\omega^2\,\mathrm{d}\omega}{c^3} \tag{5.1}$$

(Remember the usual trick with the "quantisation box" where modes are easy to count. We consider, of course, the "thermodynamic limit" where the size of the box is much larger than the thermal wavelength.)

(2) You probably have learned in thermodynamics that for Bosons (particles with integer spin), the average occupation number of a mode with energy E is given by the Bose-Einstein distribution

$$\bar{n}(E) = \frac{1}{\mathrm{e}^{\beta E} - 1}\,, \qquad \beta = \frac{1}{k_B T} \tag{5.2}$$

A quite general way to derive this form is to start from the so-called Kubo–Martin–Schwinger relation

$$\langle a_k^\dagger a_k \rangle_T = \mathrm{e}^{-\beta E_k} \langle a_k a_k^\dagger \rangle_T \tag{5.3}$$

where the operator a_k destroys one particle with the energy E_k. Use the commutation relation for the creation and annihilation operators $a_k^\dagger$, a_k (for a normalisable mode, otherwise Eq.(5.3) does not make sense) and derive Eq.(5.2) for the mean occupation number.

(3) Justify in words why the following combination of the mode density (5.1) and the mean photon number (5.2) gives the blackbody radiation spectrum

$$S(\omega)\frac{\mathrm{d}\omega}{2\pi} = \hbar\omega\,\bar{n}(\omega)\rho(\omega)\frac{\mathrm{d}\omega}{2\pi} \tag{5.4}$$

and compute this spectrum at room temperature for (i) the frequency used by your mobile phone and (ii) yellow light. (Check your units: you should get $\mathrm{J/(m^3\,Hz)}$.)

At which frequency (in THz or in cm^{-1}) occurs the peak of the spectrum? Compare to one possible definition of the thermal wavelength: $\lambda_T = \hbar c/(k_B T)$.

Problem 5.2 – The energy density of the empty Universe, qualitatively (10 points + 3 bonus points)

(1) In the energy of the quantized radiation field, we often "neglect" the contribution of the zero-point energy of each field mode, given by $\frac{1}{2}\hbar\omega_k$ per mode. Show that this "small term" leads, when integrating over all modes of the electromagnetic field, to an energy density that depends on the cutoff wavenumber k_c as follows

$$u_{\mathrm{vac}} = C\,\hbar c\,k_c^4 \tag{5.5}$$

where C is a numerical constant of order unity. [3 Bonus points: compute C.]

(2) Develop the following business model for "solving" the energy crisis: advertise that with advanced optical technologies, one can "tap" the vacuum energy density in the visible range (≈ 400–$710\,\mathrm{nm}$). You want to sell bottles (and a laser, say) for a few EUR that contain a volume V of vacuum whose energy is larger than the electricity you can buy for the same price. How large should the bottles be?

(3) Fix the cutoff for the vacuum energy at the energy scale $E_{\mathrm{GUT}} = \hbar c k_c$ for the "unification" of the fundamental interactions (electroweak and strong, except gravity, look up the keyword "grand unified theory") and make an estimate for the corresponding vacuum energy density. Another candidate for the cutoff wavenumber is the inverse of the Planck length ℓ where gravity, relativity, and quantum theory merge. [Picture: "a radiation probe to measure a short distance has so much energy content that it distorts the metric of spacetime itself."] Express these two estimates for the vacuum energy density in atomic mass units (times c^2) per cubic meter. Compare to the observed mass density in the Universe and the density of "dark energy" (responsible for the accelerated expansion of the Universe, Nobel prize in physics 2011). [Jargon: "wrongest formula in all physics."]

For an informal exposition of this so-called "cosmological constant problem", see Adler, Casey, and Jacob [*Am. J. Phys.* **63** (1995) 620]. For a recent attempt to solve it, see Wang, Zhu, and Unruh [*Phys. Rev. D* **95** (2017) 103504 = arxiv:1703.00543].