Einführung in die Quantenoptik I

Wintersemester 2019/20
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Problem Set #6 Hand out: 17 December 2019
Hand in: 07 January 2020

Problem 6.1 – Alice's ignorance and maximal entanglement (7 points)

In the currently very active field of quantum information, entanglement is a valuable "resource". The typical setting is that some source emits a pair of entangled particles one of which is sent to "Alice", the other one to "Bob". We assume that each particle is a photon that encodes via its polarisation states $|H\rangle$, $|V\rangle$ one qubit. We assume for the photon pair the state

$$|\Psi\rangle = \alpha|H, V\rangle + \beta|V, H\rangle \tag{6.1}$$

(1) Describe in words why this state is "entangled" and what kind of correlations Alice and Bob will observe when they measure and compare the polarisation of "their" photons.

(2) Such a state is called "maximally entangled" when the reduced density operator ρ_A for Alice's measurements has the largest possible entropy. The matrix elements of the reduced density operator are defined by the formula

$$\langle a|\rho_A|a'\rangle = \sum_b \langle a, b|\rho|a', b\rangle = \sum_b \langle a, b|\Psi\rangle\langle\Psi|a', b\rangle \tag{6.2}$$

Compute ρ_A from this formula and describe in words what the entropy $S(\rho_A)$ describes physically. (Similar calculations can be made for ρ_B.) Give two sets of parameters for maximally and "minimally" entangled states and try to describe their physical meaning.

(3) A state with very peculiar correlations is the "GHZ state" (Greenberger–Horne–Zeilinger)

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}|H, H, H\rangle + \frac{1}{\sqrt{2}}|V, V, V\rangle \tag{6.3}$$

Here, three qubits are shared among three "parties" (Alice, Bob and Charly). Compute the reduced state ρ_{AB} when Charly's qubit is "traced out" (i.e., that Alice and Bob ignore his measurement results).

Problem 6.2 – The Sun and us (7 points)

(1) The total energy density of thermal radiation is given by the integral over the Planck spectrum,

$$u(T) = \frac{8\pi}{(2\pi)^2} \int_0^\infty \frac{d\omega}{2\pi} \frac{\hbar\omega^3}{e^{\beta E} - 1} \tag{6.4}$$

Show that this is proportional to T^4: in order of magnitude, $u \sim k_B T/\lambda_T^3$ with the thermal wavelength $\lambda_T = \hbar c/(k_B T)$.

(2) The Sun is approximately a black body (!) at the temperature $T = 5778\,\mathrm{K}$. Compute which fraction of the spectrum falls into the range of visible light (between $400\,\mathrm{nm}$ and $710\,\mathrm{nm}$ in wavelength).

(3) Energy and entropy flux ...

Convince yourself that the energy $\Delta\dot{q}$ the Earth receives from the Sun by radiation (per day, per area) is exactly balanced by the (thermal) radiation of the Earth. What is not exactly balanced is the entropy $\Delta\dot{s} = \Delta\dot{q}/T$ (Clausius, written as a current density). Look up the keyword 'solar constant' and estimate the amount of entropy the Earth is losing into space (per day, per area). Normalized to $k_B \log 2$, this is the amount of information the Earth is receiving from the Sun (per day, per area). Compare with the amount of information in the genetic code of the biomass on Earth and estimate a time scale for the 'creation' of this information due to Sunlight. (Attention, this is a controversial subject.)

Boltzmann had already noted this in 1875: "The general struggle for existence of animate beings is not a struggle for raw materials—these, for organisms, are air, water and soil, all abundantly available—nor for energy which exists in plenty in any body in the form of heat, but a struggle for [negative] entropy, which becomes available through the transition of energy from the hot sun to the cold earth" [in "The Second Law of Thermodynamics", quoted by H. M. Nussenzveig, *Phys. Scr.* **91** (2015) 118001].

Problem 6.3 – Transverse and longitudinal fields and sources (7 points)

(1) In the lecture, we have seen an expansion of the electric and magnetic field operators into 'transverse modes'. What does this name mean? Why do we need something more when a charge density $\rho(x)$ is present?

(2) Remember the scalar potential ('C' for 'Coulomb potential')

$$\phi_C(\mathbf{x}, t) = \frac{1}{4\pi\varepsilon_0} \int dV' \frac{\rho(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} \tag{6.5}$$

that solves the Poisson equation, $-\varepsilon_0 \nabla^2 \phi_C = \rho$. Show (from classical electrodynamics) that the shifted electric field

$$\mathbf{E}_\perp = \mathbf{E} + \nabla\phi_C \tag{6.6}$$

is transverse. It solves with the magnetic field $\mathbf{H}$ inhomogeneous Maxwell equations with a source term (a current density $\mathbf{j}_\perp$)

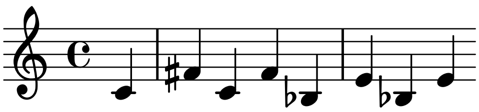
$$\begin{aligned} \varepsilon_0 \partial_t \mathbf{E}_\perp &= \nabla \times \mathbf{H} - \mathbf{j}_\perp \\ \mu_0 \partial_t \mathbf{H} &= -\nabla \times \mathbf{E}_\perp \end{aligned} \tag{6.7}$$

Give a formula for $\mathbf{j}_\perp$ and check that it is *transverse*, too, i.e., $\nabla \cdot \mathbf{j}_\perp = 0$.

(3) When these equations are used for the field operators, this scheme is often called ‘quantum electrodynamics in the Coulomb gauge’. It is convenient because the scalar potential (‘longitudinal field’) is a degree of freedom ‘enslaved’ to the matter variables (the charge density $\rho(x)$). It is inconvenient because the potential $\phi_C(x)$ is non-retarded: it depends on the instantaneous value of $\rho(\mathbf{x}', t)$ even at arbitrarily large distances $\mathbf{x} - \mathbf{x}'$. Formulate an argument why for typical atomic and molecular systems, the choice of the Coulomb gauge is nevertheless a good idea.

Problem 6.4 – Maria mit dem Kind (4 points)

Sie stehen an der Straße, und Maria fährt auf dem Weg zur Geburtshilfe vorbei:



Wie schnell fährt die Ambulanz?